

Unifying Discriminative Visual Codebook Generation with Classifier Training for Object Category Recognition (Supplemental Material for CVPR 2008 paper)

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Appendix A : Proof of Lemma 1

Proof. $l'(X_i, y_i)$ can be upper bounded by follows:

$$\begin{aligned}
 l'(X_i, y_i) &= \frac{n_i}{\sum_{j=1}^{n_i} e^{l'(\mathbf{x}_{i,j}, y_i)}} \\
 &= \frac{n_i}{\sum_{j=1}^{n_i} e(\mathbf{x}_{i,j}, y_i) \frac{\exp(\alpha g(\mathbf{x}_{i,j}, y_i))}{\sum_{y=1}^m \exp(\alpha g(\mathbf{x}_{i,j}, y))}} \\
 &= l(X_i, y_i) \frac{1}{\sum_{j=1}^{n_i} q_{i,j}(y) \frac{\exp(\alpha g(\mathbf{x}_{i,j}, y_i))}{\sum_{y=1}^m \exp(\alpha g(\mathbf{x}_{i,j}, y))}} \\
 &\leq l(X_i, y_i) \sum_{j=1}^{n_i} q_{i,j}(y) \left(\frac{\exp(\alpha g(\mathbf{x}_{i,j}, y_i))}{\sum_{y=1}^m \exp(\alpha g(\mathbf{x}_{i,j}, y))} \right)^{-1} \\
 &= l(X_i, y_i) \sum_{j=1}^{n_i} q_{i,j}(y) \sum_{y=1}^m \exp \left(\alpha (g(\mathbf{x}_{i,j}, y) - g(\mathbf{x}_{i,j}, y_i)) \right),
 \end{aligned}$$

where

$$q_{i,j}(y) = \frac{e(\mathbf{x}_{i,j}, y_i)}{\sum_{j'=1}^{n_i} e(\mathbf{x}_{i,j'}, y_i)}.$$

The above inequality gives us the result in Lemma 1. \square

Appendix B : Proof of Lemma 2

Proof. $\exp(\alpha(g(\mathbf{x}_{i,j}, y) - g(\mathbf{x}_{i,j}, y')))$ can be upper bounded as follows:

$$\begin{aligned}
 &\exp(\alpha(g(\mathbf{x}_{i,j}, y) - g(\mathbf{x}_{i,j}, y'))) \\
 &= \exp \left(-3\alpha \times \frac{g(\mathbf{x}_{i,j}, y') - g(\mathbf{x}_{i,j}, y) + 1}{3} + 3\alpha \times \frac{1}{3} \right. \\
 &\quad \left. + 0 \times \frac{g(\mathbf{x}_{i,j}, y) - g(\mathbf{x}_{i,j}, y') + 1}{3} \right) \\
 &\leq \frac{g(\mathbf{x}_{i,j}, y') - g(\mathbf{x}_{i,j}, y) + 1}{3} \exp(-3\alpha) \\
 &\quad + \frac{1}{3} \exp(3\alpha) + \frac{g(\mathbf{x}_{i,j}, y) - g(\mathbf{x}_{i,j}, y') + 1}{3} \\
 &= \frac{\exp(-3\alpha) + \exp(3\alpha) + 1}{3} \\
 &\quad - \frac{1 - \exp(-3\alpha)}{3} (g(\mathbf{x}_{i,j}, y') - g(\mathbf{x}_{i,j}, y)).
 \end{aligned}$$

In the above, we exploit the convexity of exponential function, namely

$$\exp \left(\sum_{i=1}^n p_i q_i \right) \leq \sum_{i=1}^n p_i \exp(q_i),$$

when $p_i, i = 1, \dots, n$ is a probability distribution (i.e., $p_i \geq 0, i = 1, \dots, n$ and $\sum_{i=1}^n p_i = 1$). \square

Appendix C : Proof of Theorem 1

Proof. Using the results in Lemma 1 and 2, we have the loss function $l'(X_i, y)$ upper bound as follows:

$$\begin{aligned}
& \frac{l'(X_i, y)}{l(X_i, y)} \\
& \leq \frac{\exp(-3\alpha) + \exp(3\alpha) + 1}{3} \sum_{j=1}^{n_i} \sum_{y'=1}^m q_{i,j}(y) e(X_i, y') \\
& \quad - \frac{1 - \exp(-3\alpha)}{3} \sum_{j=1}^{n_i} q_{i,j}(y) \sum_{y'=1}^m e(X_i, y') g(\mathbf{x}_{i,j}, y') \\
& \quad - \frac{1 - \exp(-3\alpha)}{3} \sum_{j=1}^{n_i} q_{i,j}(y) \sum_{y'=1}^m e(X_i, y') g(\mathbf{x}_{i,j}, y') \\
& = \frac{\exp(-3\alpha) + \exp(3\alpha) + 1}{3} \\
& \quad - \frac{1 - \exp(-3\alpha)}{3} \sum_{j=1}^{n_i} q_{i,j}(y) \sum_{y'=1}^m \delta(y, y') g(\mathbf{x}_{i,j}, y') \\
& \quad - \frac{1 - \exp(-3\alpha)}{3} \sum_{j=1}^{n_i} q_{i,j}(y) \sum_{y'=1}^m e(X_i, y') g(\mathbf{x}_{i,j}, y').
\end{aligned}$$

In the above, we use the relationships

$$\sum_{j=1}^{n_i} q_{i,j}(y) = 1, \quad \text{and} \quad \sum_{y=1}^m e(X_i, y) = 1.$$

Using the above inequality and the fact that $\mathcal{L}' = \sum_{i=1}^N l \sum_{y \in \mathbf{y}_i} l(X_i, y)$, we obtain the result in Theorem 1. \square

Appendix D : Computing the Optimal α

Proof. First, using Lemma 1, we have the following upper bound for the objective function \mathcal{L}' , i.e.

$$\begin{aligned}
\mathcal{L}' & \leq \sum_{i=1}^N \sum_{y \in \mathbf{y}_i} l(X_i, y) \sum_{j=1}^{n_i} q_{i,j}(y) \sum_{y'=1}^m \frac{e(\mathbf{x}_{i,j}, y')}{\exp(\alpha[g(\mathbf{x}_{i,j}, y) - g(\mathbf{x}_{i,j}, y')])} \quad \text{where} \\
& = \sum_{i=1}^N \sum_{y \in \mathbf{y}_i} \sum_{j=1}^{n_i} \sum_{y'=1}^m t_{i,j}(y, y') \delta(g(\mathbf{x}_{i,j}, y), 1) \delta(g(\mathbf{x}_{i,j}, y'), 0) \exp(\alpha) \\
& + \sum_{i=1}^N \sum_{y \in \mathbf{y}_i} \sum_{j=1}^{n_i} \sum_{y'=1}^m t_{i,j}(y, y') \delta(g(\mathbf{x}_{i,j}, y), 0) \delta(g(\mathbf{x}_{i,j}, y'), 1) \exp(\alpha) \quad \text{Hence,}
\end{aligned}$$

where

$$t_{i,j}(y, y') = l(X_i, y) q_{i,j}(y) e(\mathbf{x}_{i,j}, y').$$

By setting the derivative of the above expression with respect to α to be zero, we have the expression for computing the optimal α . \square

Appendix E : Proof of Theorem 3

Proof. First note that \mathcal{L} can also be written as

$$\begin{aligned}
\mathcal{L} & = \sum_{i=1}^N \sum_{y \in \mathbf{y}_i} \sum_{j=1}^{n_i} \sum_{y'=1}^m l(X_i, y) q_{i,j}(y) e(\mathbf{x}_{i,j}, y') \\
& = \sum_{i=1}^N \sum_{j=1}^{n_i} \sum_{z=0}^1 \sum_{z'=0}^1 A_{i,j}(z) B_{i,j}(z') \\
& = K_{0,0} + K_{0,1} + K_{1,0} + K_{1,1},
\end{aligned}$$

where

$$K_{z,z'} = \sum_{i=1}^N \sum_{j=1}^{n_i} A_{i,j}(z) B_{i,j}(z').$$

On the other hand, using the result for α , we have \mathcal{L}' expressed as follows:

$$\begin{aligned}
\mathcal{L}' & = 2 \sqrt{\left(\sum_{i=1}^N \sum_{j=1}^{n_i} A_{i,j}(1) B_{i,j}(0) \right) \left(\sum_{i=1}^N \sum_{j=1}^{n_i} A_{i,j}(0) B_{i,j}(1) \right)} \\
& = 2 \sqrt{K_{1,0} K_{0,1}}.
\end{aligned}$$

Hence, the ratio \mathcal{L}'/\mathcal{L} is calculated as

$$\begin{aligned}
\frac{\mathcal{L}'}{\mathcal{L}} & = \frac{2 \sqrt{K_{1,0} K_{0,1}}}{K_{0,0} + K_{0,1} + K_{1,0} + K_{1,1}} \\
& = 1 - \frac{K_{0,0} + K_{0,1} + K_{1,0} + K_{1,1} - 2 \sqrt{K_{0,1} K_{1,0}}}{K_{0,0} + K_{0,1} + K_{1,0} + K_{1,1}} \\
& \leq 1 - \frac{K_{0,1} + K_{1,0} - 2 \sqrt{K_{0,1} K_{1,0}}}{K_{0,0} + K_{0,1} + K_{1,0} + K_{1,1}} \\
& = 1 - \frac{(\sqrt{K_{0,1}} - \sqrt{K_{1,0}})^2}{K_{0,0} + K_{0,1} + K_{1,0} + K_{1,1}} \\
& = 1 - \frac{(\exp(\alpha) - 1)^2}{1 + \exp(2\alpha) + \eta},
\end{aligned}$$

$$\begin{aligned}
\eta & = \frac{K_{1,1} K_{0,0}}{K_{1,0}} \\
& = \frac{\sum_{i=1}^N \sum_{j=1}^{n_i} \sum_{z=0}^1 A_{i,j}(z) B_{i,j}(z)}{\sum_{i=1}^N \sum_{j=1}^{n_i} A_{i,j}(1) B_{i,j}(0)}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_T & = \mathcal{L}_0 \prod_{t=1}^T \frac{\mathcal{L}_t}{\mathcal{L}_{t-1}} \\
& \leq \prod_{t=1}^T \left(1 - \frac{(\exp(\alpha_t) - 1)^2}{1 + \exp(2\alpha_t) + \eta_t} \right).
\end{aligned}$$

\square