Variable Structure Control for Parameter Uncertain Stochastic Systems with Time-Varying Delay

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***Abstract*—In this paper, the variable structure control problem for a class of uncertain stochastic systems with time-varying delay is investigated. A new concept of the subordinated reachable property of the sliding motion is introduced. The variable structure control law is then proposed to ensure that the sliding motion is subordinated reachable. And a sufficient condition for mean square asymptotical stability of the sliding motion is given. A numerical example is presented to demonstrate the effectiveness of the obtained results.**

Keywords*—Mean-square stability, Variable structure control, Uncertainty,Time-varying delay*)

I. INTRODUCTION

The variable structure control (VSC) theory has been made great progress since it was proposed in the 1950's [1-3]. Owing to its easy design, simplicity of implementation and strong robustness properties against matched disturbances, VSC is a successful control method and has been used in various applications [4-5]. Recently, the VSC approach has been extended to stochastic control systems [6-10]. In [8], by using the equivalent control method, the variable control problem of stochastic systems with time-varying delay has been researched and a sliding mode controller has been designed. However, to eliminate the stochastic term in the sliding motion equation, the rather strong constitutive hypotheses on the system's model and the constraint structure are needed. In this paper, the variable control problem of stochastic systems with time-varying delay is investigated. We show that the obtained results hold under less stringent hypotheses. And this is of primary importance for application purpose. This is essentially done by introducing a notion of the subordinated reachable property of the sliding motion, which practically describes conditions of reachable property of the sliding motion by means of the mean-square convergence of sliding manifold.

This is essentially done by introducing a notion of the subordinated reachable property of the sliding motion, which This paper is organized as follows. Section 2 describes the system model and gives main lemmas. Section 3 designs the sliding mode controller. In section 4, the sliding manifold is established and the definitions of reachability are given. Meanwhile, the sufficient conditions of the sliding motion reachability are obtained. Section 5 discusses the stability of the sliding motion. Simulation results are given in Section 6. Finally, Section 7 presents some conclusions.

For convenience, we give the following notation: $\|\bullet\|$ denotes the 2-norm of a vector or its induced matrix norm, and $\|\bullet\|_1$ denotes the 1-norm of a vector. Obviously, we have $\|\alpha\| \le \|\alpha\|_1$ for $\forall \alpha \in R^n$. For a real symmetric matrix M, $M > 0$ denotes that matrix M is positive define. I represents an identity matrix of appropriate dimensions. $\max(\bullet)$ and $\text{rank}(\bullet) =$ represent the maximum eigenvalue and the rank of a matrix, respectively. $(\Omega, \mathrm{F}, (\mathrm{F}_t)_{t\ge 0}, \mathrm{P})$ is a complete probability space with Ω the sample space, F the $\sigma-$algebra of subsets of the sample space, $(\mathrm{F_t})_{t\ge 0} c$ the natural filtration and P the probability measure. E denotes the expectation operator with respect to probability measure P.

II. THE SYSTEM MODEL DESCRIPTION AND MAIN LEMMAS

Consider the uncertain time-varying delay stochastic systems

$$\begin{aligned} \mathrm{d}x(t) = \Big[& \big(A+\Delta A(t)\big)x(t) + \big(A_d + \Delta A_d(t)\big)x\big(t-\tau(t)\big) \\ & + B\big(u(t) + f\big(x(t),t\big)\big)\Big]\mathrm{d}t + \big[Cx(t) + C_d x\big(t-\tau(t)\big)\big]\mathrm{d}\omega(t) \end{aligned} \tag{1}$$

with the initial condition

$$x(t) = \varphi(t)\ ,\ t \in [-\tau, 0] \tag{2}$$

where $x(t) \in \mathrm{R}^n$ is the state, $u(t) \in \mathrm{R}^m$ is the control input, and $\omega(t)$ is a one-dimensional Brown motion. $A \in \mathrm{R}^{n\times n}$, $A_d \in \mathrm{R}^{n\times n}$, $B \in \mathrm{R}^{n\times m}$, $C \in \mathrm{R}^{n\times n}$, $C_d \in \mathrm{R}^{n\times n}$ are known real constant matrices, $\Delta A(t) \in \mathrm{R}^{n\times n}$ and $\Delta A_d(t) \in \mathrm{R}^{n\times n}$ are unknown time-varying matrices representing system parameter uncertainties. $\tau(t)$ is the time-varying delay satisfying

$$0 < \tau(t) \le \tau < \infty \tag{3}$$

where $\tau > 0$ is the known real constant. $\varphi(t)$ is a known continuous function. $f\big(x(t),t\big) \in \mathrm{R}^n$ is an unknown nonlinear function satisfying

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$$\left\| f\left(x(t),t\right)\right\| \le \beta \left\| x(t)\right\| \tag{4}$$

where $\beta > 0$ is a known constant.

Suppose that $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ has full column rank and $\det(B_2) \ne 0$, where $B_1 \in R^{(n-m)\times m}$, $B_2 \in R^{m\times m}$. Thus we can choose an invertible matrix $T^{-1} = \begin{pmatrix} I_{n-m} & -B_1 B_2^{-1} \\ 0 & I_m \end{pmatrix}$ such that $T^{-1}B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}$. By the state transformation $Ty(t) = x(t)$, Eq. (1) can be turned into

$$\begin{aligned} \mathrm{d}y(t) = &\Big[\left(T^{-1}AT + T^{-1}\Delta A(t)T\right)y(t) \\ &+\left(T^{-1}A_dT + T^{-1}\Delta A_d(t)T\right)y\left(t-\tau(t)\right) \\ &+\begin{pmatrix} 0 \\ B_2 \end{pmatrix}\left(u(x)+f\left(T^{-1}y(t),t\right)\right)\Big]\mathrm{d}t \\ &+\left[T^{-1}CTy(t)+T^{-1}C_dT\,y\left(t-\tau(t)\right)\right]\mathrm{d}\omega(t), \end{aligned} \tag{5}$$

where $T^{-1}AT = \bar{A} = \begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$,

$$T^{-1}\Delta A(t)T = \Delta\bar{A}(t) = \begin{pmatrix} \Delta A_1(t) \\ \Delta A_2(t) \end{pmatrix},$$

$$T^{-1}A_dT = \bar{A}_d = \begin{pmatrix} \bar{A}_{d1} \\ \bar{A}_{d2} \end{pmatrix} = \begin{pmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{pmatrix},$$

$$T^{-1}\Delta A_d(t)T = \Delta\bar{A}_d(t) = \begin{pmatrix} \Delta A_{d1}(t) \\ \Delta A_{d2}(t) \end{pmatrix},$$

$$T^{-1}CT = \bar{C} = \begin{pmatrix} \bar{C}_1 \\ \bar{C}_2 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

$$T^{-1}C_dT = \bar{C}_d = \begin{pmatrix} \bar{C}_{d1} \\ \bar{C}_{d2} \end{pmatrix} = \begin{pmatrix} C_{d11} & C_{d12} \\ C_{d21} & C_{d22} \end{pmatrix},$$

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}, y\left(t-\tau(t)\right) = \begin{pmatrix} y_1\left(t-\tau(t)\right) \\ y_2\left(t-\tau(t)\right) \end{pmatrix},$$

$$\bar{A}_1, \bar{A}_{d1}, \Delta A_1(t), \Delta A_{d1}(t), \bar{C}_1, \bar{C}_{d1} \in R^{(n-m)\times n},$$

$$\bar{A}_2, \bar{A}_{d2}, \Delta A_2(t), \Delta A_{d2}(t), \bar{C}_2, \bar{C}_{d2} \in R^{m\times n},$$

$$A_{11}, A_{d11}, C_{11}, C_{d11} \in \mathrm{R}^{(n-m)\times(n-m)}, A_{12}, A_{d12}, C_{12}, C_{d12} \in \mathrm{R}^{(n-m)\times m},$$

$$A_{21}, A_{22}, A_{d22}, C_{22}, C_{d22} \in \mathrm{R}^{m\times m}, A_{d21}, C_{21}, C_{d21} \in \mathrm{R}^{m\times(n-m)},$$

$$y_1(t), y_1\left(t-\tau(t)\right) \in \mathrm{R}^{n-m}, y_2(t), y_2\left(t-\tau(t)\right) \in \mathrm{R}^{m}.$$

Further, Eq. (5) can be rewritten as

$$\begin{cases} \mathrm{d}y_1(t) = \big[A_{11}y_1(t) + A_{12}y_2(t) + \Delta A_1(t)y(t) + A_{d11}y_1\left(t-\tau(t)\right) \\ \quad + A_{d12}y_2\left(t-\tau(t)\right) + \Delta A_{d1}(t)y\left(t-\tau(t)\right)\big]\mathrm{d}t \\ \quad + \left[C_{11}y_1(t) + C_{12}y_2(t) + C_{d11}y_1\left(t-\tau(t)\right)\right. \\ \quad \left. + C_{d12}y_2\left(t-\tau(t)\right)\right]\mathrm{d}\omega(t) \\ \mathrm{d}y_2(t) = \big[A_{21}y_1(t) + A_{22}y_2(t) + \Delta A_2(t)y(t) + A_{d21}y_1\left(t-\tau(t)\right) \\ \quad + A_{d22}y_2\left(t-\tau(t)\right) + \Delta A_{d2}(t)y\left(t-\tau(t)\right) + B_2u(t) \\ \quad + B_2 f\left(Ty(t),t\right)\big]\mathrm{d}t + \left[C_{21}y_1(t) + C_{22}y_2(t)\right. \\ \quad \left. + C_{d21}y_1\left(t-\tau(t)\right) + C_{d22}y_2\left(t-\tau(t)\right)\right]\mathrm{d}\omega(t) \end{cases} \tag{6}$$

As usual, we assume that the uncertainties satisfy the following conditions:

$$\begin{pmatrix} \Delta A_1(t) & \Delta A_{d1}(t) \end{pmatrix} = E_1F_1(t)\begin{pmatrix} H_{a1} & H_{ad1} \end{pmatrix} \tag{7}$$

and

$$\begin{pmatrix} \Delta A_2(t) & \Delta A_{d2}(t) \end{pmatrix} = E_2F_2(t)\begin{pmatrix} H_{a2} & H_{ad2} \end{pmatrix}. \tag{8}$$

Suppose that matrices $E_i\ (i=1,2)$, $H_{ai}, H_{adi}\ (i=1,2)$ are known constant matrices. $F_i(t)(i=1,2)$ are unknown time-varying matrices satisfying

$$F_i^T(t)F_i(t) \le I,\ i = 1,2. \tag{9}$$

Next, we give two Lemmas.

Lemma 1[5] If $F^T(t)F(t) \le I$, then

$$2X^TF(t)Y \le X^TX + Y^TY,\ X, Y \in R^n \tag{10}$$

Lemma 2[10] Let ρ, q and γ be constant satisfying $0 < \rho < \gamma$. If

$$\dot{V}(t) \le -\gamma V(t) + \rho\left|V_t\right| + q,\ t \in \left[t_0, \beta\right), \tag{11}$$

then

$$V(t) \le \left|V_{t_0}\right|\exp\left(-\lambda(t-t_0)\right) + q\lambda^{-1},\ t \in \left[t_0, \beta\right), \tag{12}$$

where

$$V(t) \in C\left(\left[t_0-\tau, \beta\right], \mathrm{R}^+\right),\ \left|V_t\right| = \sup_{\theta\in[-\tau,0]} V(t+\theta)$$

and λ is unique position solution of equation $\lambda = \gamma - \rho e^{\lambda\tau}$.

III. The Design of The Sliding Mode Controller

Select matrix $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix} \in R^{m\times n}$ with $K_1 \in R^{m\times(n-m)}$ and $K_2 \in R^{m\times m}$. Suppose that $\det(K_2) \ne 0$,

$$\mathrm{rank}\left(K^T\right) = \mathrm{rank}\left(K^T \middle| H_{ai}^T\right)$$

and

$$\mathrm{rank}\left(K^T\right) = \mathrm{rank}\left(K^T \middle| H_{adi}^T\right),$$

where $i = 1,2$. Then it is easily to select matrices $\bar{H}_{ai}$ and $\bar{H}_{adi} \in \mathrm{R}^{(n-m)\times m} \in \mathrm{R}^{(n-m)\times m}$ satisfying

$$H_{ai} = \bar{H}_{ai}K \text{ and } H_{adi} = \bar{H}_{adi}K, \tag{13}$$

where $i=1,2$. We select the switch function as follows:

$$S(t)=Ky(t)=K_1y_1(t)+K_2y_2(t). \quad (14)$$

So it follows

$$y_2(t)=K_2^{-1}S(t)-K_2^{-1}K_1y_1(t). \quad (15)$$

We design the variable structure control law of system (1) as follows:

$$\begin{aligned}u(t)=&-(K_2B_2)^{-1}[KT^{-1}Ax(t)+KT^{-1}A_dx(t-\tau(t))\\&+\beta\|K_2B_2\|\|x(t)\|\mathrm{sgn}S(t)+kS(t)+\varepsilon\,\mathrm{sgn}(S(t))]\\&-(K_2B_2)^{-1}\frac{S(t)}{2\|S(t)\|^2}\Big(2\|E_1^TK_1^TS(t)\|^2+\|\overline{H}_{a1}S(t)\|^2+\|\overline{H}_{a2}S(t)\|^2\\&+2\|E_2^TK_2^TS(t)\|^2\Big)-(K_2B_2)^{-1}\frac{1}{2}\frac{S(t)}{\|S(t)\|^2}\Big(\|\overline{H}_{ad1}S(t-\tau(t))\|^2\\&+\|\overline{H}_{ad2}S(t-\tau(t))\|^2+u_m(t),\end{aligned} \quad (16)$$

where $k>0$, $\varepsilon=\mathrm{diag}(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_m)>0$,

$$S(t)=[S_1(t),S_2(t),\ldots,S_m(t)]^T,$$

$$\mathrm{sgn}\,S(t)=[\mathrm{sgn}\,S_1(t),\mathrm{sgn}\,S_2(t),\cdots,\mathrm{sgn}\,S_m(t)]^T,$$

and

$$u_m(t)=-\frac{(K_2B_2)^{-1}S(t)}{\|S(t)\|^2}\Big(2\|K\overline{C}T^{-1}x(t)\|^2+2\|K\overline{C}_dT^{-1}x(t-\tau(t))\|^2\Big). \quad (17)$$

IV. THE SLIDING MODE REACHABILITY

Select the sliding manifold of the system (1) to be $S(t)=0$. For the system (1), the motion on the sliding manifold $S(t)=0$ is called as sliding motion. In order to describe the arrival extent to sliding motion of the system (1), we introduce the following definition:

Definition 1 Consider variable structure control systems (1). If for the state trajectory $x(t,t_0,x_0)$ tracking any place $(t_0,x_0)\in R^+\times R^n$, there always exists finite time $\overline{T}>0$ such that $\mathbb{E}(\|S(t)\|)=0$ and $\mathbb{E}(\|S(t)\|^2)=0$ when $t>\overline{T}+t_0$, then we call the sliding motion of the system (1) to be reachable. If there exists finite time $\overline{T}>0$ such that $\mathbb{E}(\|S(t)\|)=0$ and $\mathbb{E}(\|S(t)\|^2)\to 0\,(t\to+\infty)$, $t>\overline{T}+t_0$, then we call the sliding motion the system (1) to be subordinated reachable.

Theorem 1 For the variable structure control stochastic system (1), its sliding motion is subordinated reachable.

Proof Suppose that the state trajectory $x(t)$ tracks any place $(t_0,x_0)\in R^+\times R^n$.. From (6) and (14) it follows

$$\begin{aligned}\mathrm{d}S(t)=&K_1\mathrm{d}y_1(t)+K_2\mathrm{d}y_2(t)\\=&[K_1(A_{11}y_1(t)+A_{12}y_2(t))+K_2(A_{21}y_1(t)+A_{22}y_2(t))+K_1\Delta A_1y(t)\\&+K_2\Delta A_2y(t)+K_1(A_{d11}y_1(t-\tau(t))+A_{d12}y_2(t-\tau(t)))\\&+K_2(A_{d21}y_1(t-\tau(t))+A_{d22}y_2(t-\tau(t)))+K_1\Delta A_{d1}(t)y(t-\tau(t))\\&+K_2\Delta A_{d2}(t)y(t-\tau(t))+K_2B_2u(t)+K_2B_2f(Ty(t),t)]\mathrm{d}t\\&+[K_1(C_{11}y_1(t)+C_{12}y_2(t))+K_2(C_{21}y_1(t)+C_{22}y_2(t))\\&+K_1(C_{d11}y_1(t-\tau(t))+C_{d12}y_2(t-\tau(t)))+K_2(C_{d21}y_1(t-\tau(t))\\&+C_{d22}y_2(t-\tau(t)))]\mathrm{d}\omega(t).\end{aligned}$$

Denote that $\Psi_1(S(t))=\|S(t)\|=\sqrt{S^T(t)S(t)}$ ⊳ψ Using Itô formula, one obtains

$$\begin{aligned}\mathrm{L}\Psi_1(S(t))=&\frac{1}{\|S(t)\|}S^T(t)[K_1(A_{11}y_1(t)+A_{12}y_2(t))\\&+K_2(A_{21}y_1(t)+A_{22}y_2(t))+K_1\Delta A_1(t)y(t)+K_2\Delta A_2(t)y(t)\\&+K_1(A_{d11}y_1(t-\tau(t))+A_{d12}y_2(t-\tau(t)))+K_2(A_{d21}y_1(t-\tau(t))\\&+A_{d22}y_2(t-\tau(t)))+K_1\Delta A_{d1}(t)y(t-\tau(t))\\&+K_2\Delta A_{d2}(t)y(t-\tau(t))+K_2B_2u(t)+K_2B_2f(Ty(t),t)]\\&+[K_1(C_{11}y_1(t)+A_{12}y_2(t))+K_2(C_{21}y_1(t)+C_{22}y_2(t))\\&+K_1(C_{d11}y_1(t-\tau(t))+C_{d12}y_2(t-\tau(t)))\\&+K_2(C_{d21}y_1(t-\tau(t))+C_{d22}y_2(t-\tau(t)))]^T\\&\left(\frac{I_m}{\|S(t)\|}-\frac{S(t)S^T(t)}{\|S(t)\|^3}\right)[K_1(C_{11}y_1(t)+C_{12}y_2(t))\\&+K_2(C_{21}y_1(t)+C_{22}y_2(t))+K_1(C_{d11}y_1(t-\tau(t))\\&+C_{d12}y_2(t-\tau(t)))+K_2(C_{d21}y_1(t-\tau(t))+C_{d22}y_2(t-\tau(t)))].\end{aligned}$$

From (7), (8) and (13), one obtains

$$S^T(t)K_1\Delta A_1y(t)=S^T(t)K_1E_1F_1(t)\overline{H}_{a1}S(t),$$

$$S^T(t)K_2\Delta A_2y(t)=S^T(t)K_2E_2F_2(t)\overline{H}_{a2}S(t),$$

$$S^T(t)K_1\Delta A_{d1}(t)y(t-\tau(t))=S^T(t)K_1E_1F_1(t)\overline{H}_{ad1}S(t-\tau(t)),$$

and $S^T(t)K_2\Delta A_{d2}(t)y(t-\tau(t))$

$$=S^T(t)K_2E_2F_2(t)\overline{H}_{ad2}(t)S(t-\tau(t)).$$

Based on (9) and Lemma 1, one obtains

$$\begin{aligned}S^T(t)K_1E_1F_1(t)\overline{H}_{a1}S(t)\le&\frac{1}{2}S^T(t)K_1E_1E_1^TK_1^TS(t)\\&+\frac{1}{2}S^T(t)\overline{H}_{a1}^T\overline{H}_{a1}S(t),\end{aligned}$$

$$\begin{aligned}S^T(t)K_2E_2F_2(t)\overline{H}_{a2}S(t)\le&\frac{1}{2}S^T(t)K_2E_2E_2^TK_2^TS(t)\\&+\frac{1}{2}S^T(t)\overline{H}_{a2}^T\overline{H}_{a2}S(t),\end{aligned}$$

$$S^T(t)K_1E_1F_1(t)\overline{H}_{ad1}S(t-\tau(t))\le \frac{1}{2}S^T(t)K_1E_1E_1^TK_1^TS(t) + \frac{1}{2}S^T(t-\tau(t))\overline{H}_{ad1}^T\overline{H}_{ad1}S(t-\tau(t))$$

and

$$S^T(t)K_2E_2F_2(t)\overline{H}_{ad2}(t)S(t-\tau(t))\le \frac{1}{2}S^T(t)K_2E_2E_2^TK_2^TS(t) + \frac{1}{2}S^T(t-\tau(t))\overline{H}_{ad2}^T\overline{H}_{ad2}S(t-\tau(t)).$$

Further from (16) and (17), one obtains

$$\begin{aligned}\mathrm{L}\Psi_1(S(t)) &\le -\frac{1}{\|S(t)\|}S^T(t)\varepsilon\,\mathrm{sgn}(S(t))+\frac{1}{\|S(t)\|}S^T(t)K_2B_2u_m(t)\\ &+\frac{1}{\|S(t)\|}\left[K\overline{C}y(t)+K\overline{C}_dy(t-\tau(t))\right]^T\left[K\overline{C}y(t)+K\overline{C}_dy(t-\tau(t))\right]\\ &\le -\frac{1}{\|S(t)\|}S^T(t)\varepsilon\,\mathrm{sgn}(S(t))\le -\varepsilon_{\min},\end{aligned}$$

where $\varepsilon_{\min}=\min\{\varepsilon_i, i=1,2,\ldots,m\}$. Using Itô formula obtains $(\mathrm{E}\Psi_1(S(t)))'=\mathrm{EL}(\Psi_1(S(t)))\le -\varepsilon_{\min}$, which implies that

$$\mathrm{E}\Psi_1(S(t))-\mathrm{E}\Psi_1(S(t_0))\le -\varepsilon_{\min}(t-t_0).$$

Then we can select a constant

$$t_f\le \mathrm{E}\|S(t_0)\|/\varepsilon_{\min}\le \|KT\|\mathrm{E}\|\varphi(0)\|/\varepsilon_{\min}$$

such that

$$\mathrm{E}\|S(t)\|=\mathrm{E}(\Psi_1(S(t)))=0,\ t\ge t_0+t_f.$$

Denote that $\Psi_2(S(t))=S^T(t)S(t)$. Similarly, we can easily obtain

$$\mathrm{L}\Psi_2(S(t))\le -k\Psi_2(S(t))$$

and

$$\lim_{t\to+\infty}\mathrm{E}\left(\|S(t)\|^2\right)=\lim_{t\to+\infty}\mathrm{E}(\Psi_2(S(t)))=0.$$

The proof is complete.

V. STABILITY OF THE SLIDING MOTION

Assume that P is the positive solution of the matrix equation

$$\widehat{A}_1^TP+P\widehat{A}_1=-I_{n-m},$$

where $\widehat{A}_1=A_{11}+A_{d11}-(A_{12}+A_{d12})K_2^{-1}K_1$. Denote that

$$\gamma=\left(1-3\left\|P\left(A_{d11}-A_{d12}K_2^{-1}K_1\right)\right\|-\left\|PA_{12}K_2^{-1}\right\|-\left\|PA_{d12}K_2^{-1}\right\| -2\left\|E_1^TP\right\|^2-4\|P\|\left\|C_{11}-C_{12}K_2^{-1}K_1\right\|^2\right)\Big/\lambda_{\max}(P), \tag{18}$$

$$\mu=\frac{1}{\lambda_{\min}(P)}\left(\left\|P\left(A_{d11}-A_{d12}K_2^{-1}K_1\right)\right\| +4\|P\|\left\|C_{d11}-C_{d12}K_2^{-1}K_1\right\|^2\right), \tag{19}$$

$$\eta=\left\|PA_{12}K_2^{-1}\right\|+\left\|\overline{H}_{a1}\right\|^2+4\|P\|\left\|C_{12}K_2^{-1}\right\|^2 +\left\|PA_{d12}K_2^{-1}\right\|+\left\|\overline{H}_{ad1}\right\|^2+4\|P\|\left\|C_{d12}K_2^{-1}\right\|^2. \tag{20}$$

Definition 2 If for each $\varepsilon>0$ there exists a constant $\delta(\varepsilon)>0$ such that $\sup_{-\tau\le t\le 0}\mathrm{E}\|\varphi(t)\|^2<\delta(\varepsilon)$, $\mathrm{E}\|x(t,\varphi)\|^2<\varepsilon$, $t>0$, then the system (1) is said to be mean-square stable at the equilibrium.

Definition 3 If the systems (1) is said to be mean-square stable at the equilibrium and $\lim_{t\to+\infty}\mathrm{E}\|x(t,\varphi)\|^2=0$, then the system (1) is said to be asymptotically mean-square stable at the equilibrium.

If the sliding motion of systems (1) is reachable (or subordinated reachable), one obtains $\mathrm{E}\|S(t)\|=0$ based on the definition (1). Obviously, the motions confined in the manifold $\mathrm{E}\|S(t)\|=0$ are described by (1). (1) is called as the sliding motion equation. Here we can see that the sliding motion equation (1) contains the stochastic term. Next we study the stability of sliding motion equation (1).

Theorem 2 If the sliding motion of system (1) is subordinated reachable and $0<\mu<\gamma$, then the sliding motion of variable structure control systems (1) is asymptotically mean-square stable at the equilibrium.

Proof Substituting (15) into the first form of (6) one obtains

$$\begin{aligned}\mathrm{d}y_1(t)&=\left[\left(A_{11}-A_{12}K_2^{-1}K_1\right)y_1(t)+A_{12}K_2^{-1}S(t)\right.\\ &\left.+A_{d12}K_2^{-1}S(t-\tau(t))+\Delta A_{d1}(t)y(t-\tau(t))\right]\mathrm{d}t\\ &+\left[\left(C_{11}-C_{12}K_2^{-1}K_1\right)y_1(t)+C_{12}K_2^{-1}S(t)\right.\\ &\left.+\left(C_{d11}-C_{d12}K_2^{-1}K_1\right)y_1(t-\tau(t))+C_{d12}K_2^{-1}S(t-\tau(t))\right]\mathrm{d}\omega(t).\end{aligned}$$

Select the following Lyapunov function

$$\Psi(y_1(t))=y_1^T(t)Py_1(t).$$

Using Itô formula, one obtains

$$\begin{aligned}\mathrm{L}\Psi(y_1(t))&=-y_1^T(t)y_1(t)+2y_1^T(t)P\left(A_{d11}-A_{d12}K_2^{-1}K_1\right)\\ &\left(y_1(t-\tau(t))-y_1(t)\right)+2y_1^T(t)P\left[A_{12}K_2^{-1}S(t)+\Delta A_1(t)y(t)\right.\\ &\left.+A_{d12}K_2^{-1}S(t-\tau(t))+\Delta A_{d1}(t)y(t-\tau(t))\right]+\left[\left(C_{11}-C_{12}K_2^{-1}K_1\right)y_1(t)\right.\\ &+C_{12}K_2^{-1}S(t)+\left(C_{d11}-C_{d12}K_2^{-1}K_1\right)y_1(t-\tau(t))\\ &\left.+C_{d12}K_2^{-1}S(t-\tau(t))\right]^{\mathrm{T}}P\left[\left(C_{11}-C_{12}K_2^{-1}K_1\right)y_1(t)+C_{12}K_2^{-1}S(t)\right.\\ &\left.+\left(C_{d11}-C_{d12}K_2^{-1}K_1\right)y_1(t-\tau(t))+C_{d12}K_2^{-1}S(t-\tau(t))\right].\end{aligned}$$

Using (7), (9) and Lemma 1, one obtains

$$
\begin{aligned}
2y_1^T(t)P\Delta A_1(t)y(t) &= 2y_1^T(t)PE_1F_1(t)H_{a1}y(t) \\
&\le y_1^T(t)PE_1E_1^TP^Ty_1(t)+y^T(t)H_{a1}^TH_{a1}y(t) \\
&\le \|E_1P\|^2\|y_1(t)\|^2+\|H_{a1}\|^2\|y(t)\|^2 .
\end{aligned}
$$

Similarly, one obtains

$$2y_1^T(t)P\Delta A_{d1}(t)y(t-\tau(t))\le\|E_1P\|^2\|y_1(t)\|^2+\|H_{ad1}\|^2\|y(t-\tau(t))\|^2 .$$

Therefore, we can obtain

$$
\begin{aligned}
&L\Psi(y_1(t))\le-\gamma\Psi(y_1(t))+\mu\Psi(y_1(t-\tau(t))) \\
&+\left(\|PA_{12}K_2^{-1}\|+\|\overline{H}_{a1}\|^2+4\|P\|\|C_{12}K_2^{-1}\|^2\right)\|S(t)\|^2 \\
&+\left(\|PA_{d12}K_2^{-1}\|+\|\overline{H}_{ad1}\|^2+4\|P\|\|C_{d12}K_2^{-1}\|^2\right)\|S(t-\tau(t))\|^2 .
\end{aligned}
$$

Denote that $|E\Psi_t|=\sup_{\theta\in[-\tau,0]}E\Psi(y_1(t+\theta))$. According to the subordinated reachability of the sliding motion, we obtain $E\|S(t)\|^2\to0(t\to+\infty)$. So for $\forall\varepsilon>0$, there exists $t_f>t_0+\tau$ such that $E\|S(t)\|^2<\delta$ when $t>t_f$ .Here $\delta=\lambda\varepsilon/2\eta$, $\lambda>0$ is unique solution of the equation $\lambda=\gamma-\mu e^{\lambda\tau}$. Since $\left(E\Psi(y_1(t))\right)'=EL\Psi(y_1(t))$, one obtains

$$\left(E\Psi(y_1(t))\right)'\le-\gamma E\Psi(y_1(t))+\mu E\Psi(y_1(t-\tau(t)))+q ,$$

where $q=\dfrac{\lambda\varepsilon}{2}$ .Thus it follows from lemma 2 that

$$E\Psi(y_1(t))\le|E\Psi_T|e^{-\lambda(t-T)}+q\lambda^{-1} .$$

Select $t^*=t_f+\dfrac{1}{\lambda}\ln\dfrac{2|E\Psi_{t_f}|}{\varepsilon}$. Then it follows

$$E\Psi(y_1(t))\le\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon\ ,\ t_f>t^* .$$

Thus it follows $E\Psi(y_1(t))\to0(t\to+\infty)$, that is,

$$E\|y_1(t)\|^2\to0(t\to+\infty) .$$

Since
$$
\begin{aligned}
E\|y_2(t)\|^2 &= E\|K_2^{-1}S(t)-K_2^{-1}K_1y_1(t)\|^2 \\
&\le 2\|K_2^{-1}\|^2E\|K_2^{-1}S(t)\|^2+2\|K_2^{-1}K_1\|^2E\|y_1(t)\|^2 ,
\end{aligned}
$$

then one obtains $E\|y_2(t)\|^2\to0(t\to+\infty)$. Furthermore it follows

$$E\|y(t)\|^2\le E\|y_1(t)\|^2+E\|y_2(t)\|^2\to0(t\to+\infty) .$$

Thus $E\|x(t)\|^2\to0(t\to+\infty)$. The proof is complete.

VI. NUMERICAL EXAMPLE

Consider the uncertain stochastic systems (1) with

$$A=\begin{pmatrix}-10&1.5\\0.6&0.5\end{pmatrix},\ A_d=\begin{pmatrix}-2&1\\-3&3\end{pmatrix},$$

$$C=\begin{pmatrix}-0.2&0\\0.1&0.3\end{pmatrix},\ C_d=\begin{pmatrix}0.2&0.1\\-0.4&0.25\end{pmatrix},$$

$$\Delta A(t)=\begin{pmatrix}\frac{4}{5}\sin(7\pi t)+1.6\cos(7\pi t) & \frac{4}{15}\sin(7\pi t)+\frac{8}{15}\cos(7\pi t)\\ \frac{2}{3}\cos(13\pi t)+\frac{5}{6}\sin(13\pi t) & \frac{5}{36}\cos(13\pi t)+\frac{5}{18}\sin(13\pi t)\end{pmatrix},$$

$$\delta A_d(t)=\begin{pmatrix}\frac{6}{5}\sin(7\pi t)+\frac{12}{5}\cos(7\pi t) & \frac{2}{5}\sin(7\pi t)+\frac{4}{5}\cos(7\pi t)\\ \frac{1}{3}\cos(13\pi t)+\frac{1}{6}\sin(13\pi t) & \frac{1}{9}\cos(13\pi t)+\frac{1}{18}\sin(13\pi t)\end{pmatrix},$$

$$f(x(t),t)=\begin{pmatrix}0.3x_1(t)\\0.5x_2(t)\end{pmatrix},\ B=\begin{pmatrix}0\\1\end{pmatrix}$$

and the initial value conditions:

$$x_1(t)=(5t)^4-0.05\ ,\ x_2(t)=-(2t-0.5)^3$$

with $-0.1\le t\le0$. For convenience, we select constant delay $\tau(t)=0.1$ and the state transformation matrix $T=\begin{pmatrix}1&0\\0&1\end{pmatrix}$.

Choose $\beta=0.6$, $K=(5\quad 5/3)$, $\varepsilon=5$ and $k=0.4$. From (14) and (16), it follows that the switching function $s(t)=5x_1(t)+5/3\,x_2(t)$ and the variable structure control law

$$
\begin{aligned}
u(t) &=-0.6[-49x_1(t)+8.3333x_2(t)-15x_1(t-0.1)+10x_2(t-0.1) \\
&+\|x(t)\|\operatorname{sgn}(S(t))+0.4S(t)+0.1\operatorname{sgn}(S(t))]-12.3365S(t) \\
&-0.12\frac{S(t)}{\|S(t)\|^2}\|S(t-0.1)\|^2-\frac{1.2S(t)}{\|S(t)\|^2}(\|-0.8333x_1(t)+0.5x_2(t)\|^2 \\
&+\|0.3333x_1(t-0.1)+0.9167x_2(t-0.1)\|^2).
\end{aligned}
$$

It is easy to certify that the conditions of theorem 1, 2 are satisfied. Simulation results are provided in fig. 1-3. Fig.1 indicates the time response of variables $\|s(t)\|$. We can see that the sliding motion is substituted reachable. Fig. 2 and 3 are trajectories of state $x(t)$. It is seen that the sliding motions are asymptotically mean-square stable (in probability).

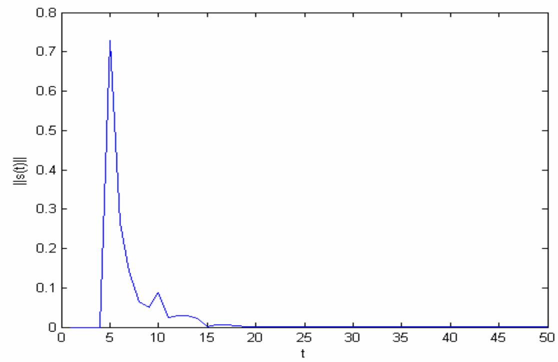

Fig. 1 Time response of variables ||S(t)||

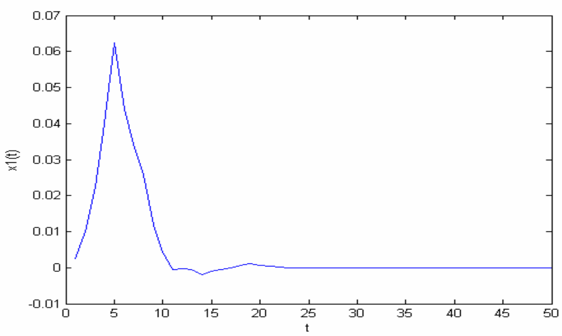

Fig. 2 Trajectories of state x1(t)

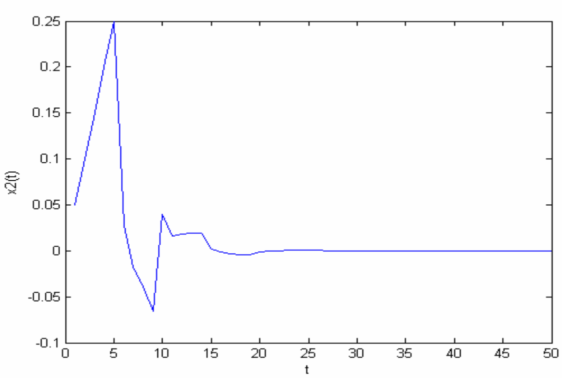

Fig.3 Trajectories of state x2(t)

VII. CONCLUSION

The variable structure control problem of a class of uncertain stochastic systems with time-varying delays is under study. A sliding mode control law has been synthesized such that the sliding motion has subordinated reachability. Furthermore, under the condition of subordinated reachability, a sufficient condition for the mean-square stability of sliding motion has been proposed. The results obtained hold under some rather less stringent constitutive hypotheses, which will be helpful to the application in engineering practice. The results obtained are supported by a numerical example.

ACKNOWLEDGMENT

This work was supported in part by China Postdoctoral Science Foundation Under Grant # 20070420368 and by the North China Electric Power University Under Grant # 200512015.

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