Near-Optimal Trajectory Planning of a Spherical Mobile Robot for Environment Exploration

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***Abstract*—Spherical mobile robot has the characteristics of compact structure and agile motion and is promising to be used in unmanned environment. Designed for environment exploration, a spherical mobile robot BHQ-2 was briefly introduced. The kinematic equation of BHQ-2 was established and its controllability was proved with controllable Lie algebra. Based on the Ritz approximation theory, the near-optimal trajectory of BHQ-2 was planned with the Gauss-Newton algorithm. Simulation results and experimental results of the spherical mobile robot to plan trajectory with the method were presented.**

***Keywords*—spherical mobile robot, nonholonomic, trajectory planning**

I. INTRODUCTION

Spherical mobile robot is a special type of mobile robot that has a ball-shaped outer shell to include all the mechanism, control devices and energy sources inside. Different from such traditional mobile robots as wheeled robot, tracked robot, spherical mobile robot realizes its motion with the principles of offset of gravity center or conservation of momentum. The special structure and driving manner make spherical mobile robot have some unique characteristics, such as compact and well-sealed structure, agile motion and never overturning. For example, a spherical mobile robot can pass through a narrow and bending area that is a little bigger than its diameter, where no traditional mobile robots can get across at present. Spherical mobile robot has the advantage to survive in such unmanned or harsh environment as outer planets, deserts and earthquake ruins, to do some exploration or reconnaissance tasks[1-6].

From the perspective of control, spherical mobile robot is a kind of nonholonomic system that can control more degrees of freedom with less drive inputs, and researchers have done some work on its motion analysis and control. Aarne Halme, Jussi Suomela, et al established the kinematic and dynamic model of a spherical mobile robot and made some studies on its open-loop path planning. Experiment results showed that the control performance of this spherical mobile robot was poor[1]. Bicchi, et al established a quasi-static kinematic model and a planar Lagrangian dynamic model of a spherical mobile robot, and realized the trajectory planning by the way of subsection input. It's shown that those models are only valid in limited conditions[2]. Shourov Bhattacharya and Sunil K. Agrawal deduced the first-order mathematical model of a kind of spherical mobile robot from the non-slip constraint and the conservation of angular momentum, and studied its path planning based on the strategy of optimal time and energy, simulations and experiment results were presented[3]. Amir Homayoun Javadi, et al developed a kind of spherical mobile robot by adjusting the weights of four spokes respectively to change the gravity center of the robot. Dynamic modelling, path planning were also studied, and motion of the robot in a small range was simulated and analyzed[5]. Zhan Qiang, Zhou Tingzhi, et al established the dynamic model of BHQ-1 with a simplified Boltzmann-Hamel equation and deduced the expressions of the input moments of two motors to drive the robot to move along straight trajectory and circular trajectory separately from the dynamic model[15]. In the aspect of optimal control of nonholonomic system, Brockett first systematically studied the optimal control of non-drift nonholonomic system, by constructing Lagrangian functions and Lagrangian equations with object functions, the conclusion that the optimal input is Sine function and elliptic function respectively was achieved[9]. Upon this result, Murray and Sastry applied the Sine function input into the control of the nonholonomic chained system[10]. Royhanoglu applied the Stokes theorem and the Taylor series expansion into the nonholonomic system and proposed a new motion planning algorithm[11]. Gurvits et al, studied the motion planning problem of the nonholonomic system based on the average theorem, and their basic idea is to move in the direction according to the Lie brackets by using cycle control inputs of high-frequency and high-amplitude, thus adjust the system to the goal position from the initial point[12]. Leomard et al also made some research on the motion planning of the nonholonomic system based on the average theorem[1-3].

As a whole, at present the research on the control of spherical mobile robot is still on the initial stage, although some researchers have done some work, there still isn't systematic theory or methods that have reliable and stable control results.

In this paper, the spherical mobile robot of BHQ-2 was introduced firstly, and then the kinematic model and controllability of this system were analyzed, after that, the near-optimal trajectory of the spherical mobile robot was planned based on Gauss-Newton method, and finally, the simulation results and experimental results were presented.

II. BRIEF INTRODUCTION OF SPHERICAL MOBILE ROBOT BHQ-2

BHQ-2 is a spherical mobile robot designed for environment exploration by our lab, which structure is shown in Fig. 1. The inner driving mechanism of the spherical robot is mainly composed of one hollow axle, one small two-wheel car, two cameras and a heavy made up of battery group. The hollow axle is connected with the spherical shell by two ball hearings on both sides, the small car is fixed to the hollow axle with its two wheels contacting with the shell in terms of rolling friction and the heavy is installed on a rotating shaft in the hollow axle. Two mini CCD cameras are installed inside the hollow axle and can extend to the outside of the shell when needed and draw back to the inside of the shell when not. The environment images taken by these cameras can be transmitted to a control center through a wireless image transmission system. According to the image an operator can not only observe the environment but also tele-operate the motion of the spherical robot by a joystick.

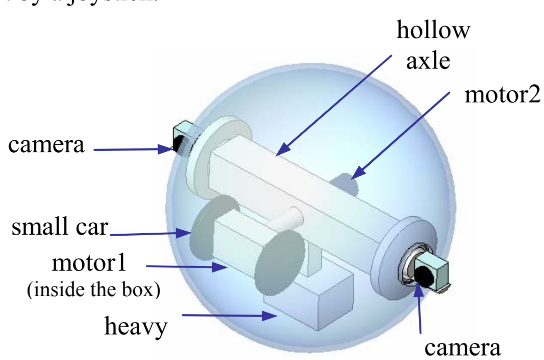

Figure 1. Structure of spherical robot BHQ-2

The main moving principle of BHQ-2 is that when motor 2 keeps still, the small car driven by motor1 will climb up the inner inside of the spherical shell so as to lift the heavy to a higher position and a driving moment caused by the displacement of the gravity center of the robot will drive the robot to move straight. When the driving moment and the rolling friction moment reach balance, the robot will move forward in a constant speed and the angle displacement between the inner mechanism and the ground will be maintained, thus the two cameras could keep a steady state and get ready for observing the environment. If motor 2 also rotate to drive the heavy to turn a certain angle, a lateral eccentric torque will be obtained and drives the robot to turn a tilted angle, thus drives the robot to turn aside.

III. KINEMATIC MODEL AND CONTROLLABILITY OF BHQ-2

Assuming spherical robot BHQ-2 rolls without slipping on a plane, its constraint conditions can be written as

$$\begin{cases} \dot{x} - r(\dot{\varphi}\cos\beta\sin\psi + \dot{\beta}\cos\psi) = 0 \\ \dot{y} - r(\dot{\varphi}\cos\beta\cos\psi - \dot{\beta}\sin\psi) = 0 \\ \omega_z = \dot{\psi} + \dot{\varphi}\sin\theta \end{cases} \tag{1}$$

Where, （ x, y ） are the coordinates of the geometric center of the robot; ψ, β, φ are the pose of the robot expressed by ZXZ Euler angle; r is the radius of the robot; ω_z is the robot's rotational speed in vertical direction; $\dot{\psi}$ is the robot's rotation speed driven by the total external force. Because there is no external force on the robot, $\dot{\psi}$ is zero, that is to say $\omega_z = \dot{\varphi}\sin\theta$. Then Eq. (1) can be changed to

$$\dot{\boldsymbol{p}} = \boldsymbol{g}_1(\boldsymbol{p})\dot{\varphi} + \boldsymbol{g}_2(\boldsymbol{p})\dot{\beta} \tag{2}$$

Where,

$$\boldsymbol{p} = (x, y, \psi, \varphi, \beta)^{\mathrm{T}}$$

$$\boldsymbol{g}_1 = \begin{bmatrix} r\cos\beta\sin\psi & -r\cos\beta\cos\psi & -\sin\beta & 1 & 0 \end{bmatrix}^T$$

$$\boldsymbol{g}_2 = \begin{bmatrix} r\cos\psi & r\sin\psi & 0 & 0 & 1 \end{bmatrix}^T$$

For a given input $u(t) \in \Re^2$, if and only if $p(t)$ satisfy the above-mentioned equation Eq.(2) , $p(t)$ is the feasible trajectory. If the system can achieve motion between any two points, then the constraint is completely nonholonomic and the system configuration is unrestricted. On the contrary, if the constraint is holonomic, then the motion of the system is constrained in an appropriate surface, and the control system can only move between two points in a given manifold [14]. That is to say, there's a certain relationship between the controllability and the property of constraint. Firstly, we give the proof of nonholonomic property of the spherical mobile robot.

$$[\boldsymbol{g}_1, \boldsymbol{g}_2] = \frac{\partial \boldsymbol{g}_2}{\partial \boldsymbol{p}}\boldsymbol{g}_1 - \frac{\partial \boldsymbol{g}_1}{\partial \boldsymbol{p}}\boldsymbol{g}_2$$

$$= \begin{bmatrix} 0 & 0 & -rs\psi & 0 & 0 \\ 0 & 0 & rc\psi & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} rc\beta s\psi \\ -rc\beta c\psi \\ -s\beta \\ 1 \\ 0 \end{bmatrix} -$$

$$\begin{bmatrix} 0 & 0 & rc\beta c\psi & 0 & -rs\beta s\psi \\ 0 & 0 & rc\beta s\psi & 0 & rs\beta c\psi \\ 0 & 0 & 0 & 0 & -c\beta \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} rc\psi \\ rs\psi \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & c\beta & 0 & 0\end{bmatrix}^T \notin \Delta \tag{3}$$

Here , g_1, g_2 are vector functions of p . So from Eq. (2) we can see that Δ , which is the differential of p , is a space spanned by $g1, g2$, that is $\Delta = span\{g_1(p), g_2(p)\}$. Now we consider the involution of vector functions $g1, g2$ using Lie Algebra

Because $[g_1, g_2] \notin \Delta$, the differential Δ is not involutive. From Frobenius theorem[10] we know that it is not integrable, so the system is a nonholonomic system.

For this nonholonomic system, let $\overline{\Delta} = L(g_1, g_2)$ be the control Lie algebra. According to the definition of Lie Bracket, we can get

$$g_3 = [g_1,[g_1,g_2]],\ g_4 = [g_1,[g_1,g_2]],\ g_5 = [g_2,[g_1,g_2]],$$

and the third order Lie algebra of distribution Δ is $L(\Delta) = span\{g_1, g_2, g_3, g_4\}$. Calculation shows that the rank of $L(\Delta)$ is 5. From CHOW's theorem[10], we know that the system is completely nonholonomic and is controllable.

From the analysis above-mentioned, we know that, the system is controllable since the Lie algebra of BHQ-2 is full rank. And we can solve the motion control problem theoretically.

IV. NEAR-OPTIMAL TRAJECTORY PLANNING

Near-optimal trajectory planning is to search a proper set of control inputs to drive the robot to the destination from a certain initial state. Spherical mobile robot includes all its energy sources inside its shell and its energy sources are limited to the size and structure of the robot, so in order to make a spherical robot to do much more tasks with the limited energy sources, time and energy based near-optimal trajectory planning is very significant. Based on the discussion above, we change Eq. (2) into the form as

$$\dot{p} = B(p)\cdot u \tag{4}$$

Where，

$$p = \begin{bmatrix}x & y & \psi & \beta\end{bmatrix}^T,\ B = \begin{bmatrix}g_1 & g_2\end{bmatrix},\ u = \begin{bmatrix}\dot{\varphi} & \dot{\beta}\end{bmatrix}^T$$

We take the energy of the spherical mobile robot as the optimization object, and then we construct the cost function of the system as

$$J = \int_0^T \langle u, u\rangle dt \tag{5}$$

Now, we can find that the near-optimal trajectory planning problem of the spherical mobile robot is equivalent to the optimized problem with one fixed point. As we have analyzed before, the spherical mobile robot system is controllable, then there must be a solution vector $u^* \in L_2([0,T])$, which will meet the minimum cost function of spherical mobile robot. Here, $L_2([0,T])$ represents the measurable vector-valued function in Hilbert space. Taking Fourier basis as the orthonormal basis of $L_2([0,T])$, denoted as $\{e_i\}_{i=1}^{\infty}$, then the input $u(t)$ can be expressed as

$$u = \sum_{i=1}^{\infty} \alpha_i e_i \tag{6}$$

Where, $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_N)^T$ is the projection of the input $u(t)$ onto the orthonormal basis. Then the cost function can be written as

$$J = \int_0^T \langle u, u\rangle dt = \sum_{i=1}^{\infty} \alpha_i^2 \tag{7}$$

Thus, the near-optimal trajectory planning problem becomes a problem of finding the solution in terms of $\alpha = (\alpha_1, \alpha_2, \cdots \alpha_N, \cdots)^T$ to meet the minimum of the cost function, and this is a nonlinear optimization problem in an infinite-dimensional space. Considering the orthogonal character of the Fourier basis, we take α as the new control variable, and bring in γ as the penalty factor, the cost function becomes

$$J = \sum_{i=1}^{\infty} \alpha_i^2 + \gamma \left\| p(T) - p_f \right\|^2 \tag{8}$$

Where, $\gamma > 0$, and $p(T) \in R^4$ is the solution of $\dot{p} = B(p)\cdot u$ at the time of $t = T$ when given the certain control input $u(t)$. Obviously, $p(T)$ is the function of α . Assuming that $p(T) = f(\alpha)$, as long as γ and N are given by the certain value, the cost function becames

$$J = \langle \alpha, \alpha\rangle + \gamma \left\| f(\alpha) - p_f \right\|^2 \tag{9}$$

Where, α is infinite-dimension and seems impossible to obtain the solution. But by applying the Ritz approximation method we can just choose its limited dimension and then obtain an approximate solution of the problem. For example, if we choose the former *N* items, that is to subject

$$u = \sum_{i=1}^{N} \alpha_i e_i \text{, Where } \alpha = (\alpha_1, \alpha_2, \cdots \alpha_N)^T$$

After transforming the equation above according to Taylor series expansion by using the popular Newton's algorithm, we choose the second-order approximation and amend it to be the Gauss - Newton iterative formula

$$\alpha_{k+1} = \alpha_k - \mu\left[\sigma I + A^T A\right]^{-1}\left[\sigma\alpha_k + A^T(f(\alpha_k) - p_f)\right] \tag{10}$$

Where, A is the Jacobian matrix of $f(\alpha)$, $A = \partial f / \partial\alpha$, $\sigma = 1/\gamma$, I is the unit matrix, μ is a step parameter and is constrained as $0 < \mu < 1$. We define by ϕ the $4 \times N$ matrix, whose column elements to be the Fourier basis elements $\{e_i(t)\}_{i=1}^{\infty}, t \in [0,T]$

$$\phi = [e_1(t), e_2(t) \cdots e_N(t)] \tag{11}$$

Then, the original system can be expressed as

$$\dot{p} = B(p) \cdot u = B(p) \cdot \phi \cdot \alpha \tag{12}$$

Let $Y(t) = \partial p(t) / \partial \alpha$, and obviously, A is the value of $Y(t)$ when $t = T$. The differential equation for $Y(t)$ will be

$$\begin{aligned}
\dot{Y}(t) &= \frac{\partial}{\partial t}\left(\frac{\partial p}{\partial \alpha}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial p}{\partial t}\right) = \frac{\partial}{\partial \alpha}\dot{p} \\
&= \frac{\partial}{\partial \alpha}\left(\sum_{i=1}^{m} B^i u_i\right) = \sum_{i=1}^{m}\left(\frac{\partial B^i}{\partial \alpha}u_i + B^i \frac{\partial u_i}{\partial \alpha}\right) \\
&\sum_{i=1}^{m}\frac{\partial B^i}{\partial \alpha}u_i + B\phi = \sum_{i=1}^{m}\left(\frac{\partial B^i}{\partial p}\frac{\partial p}{\partial \alpha}\right)u_i + B\phi \\
&= \sum_{i=1}^{m}\left(\frac{\partial B^i}{\partial p}Y\right)u_i + B\phi = \left(\sum_{i=1}^{m}\frac{\partial B^i}{\partial p}u_i\right)Y + B\phi
\end{aligned} \tag{13}$$

By combining Eq. (13) with Eq. (2), we have

$$\begin{cases} \dot{p} = B(p) \cdot \phi \alpha_k \\ \dot{Y} = \left(\displaystyle\sum_{i=1}^{m}\frac{\partial B^i}{\partial p}u_i\right)Y + B\phi \end{cases} \tag{14}$$

Now, if we set $f(\alpha_n) = p(T)$, $A = Y(T)$ and integrate the set of differential equations above from t = 0 to t = T at each time step to update α, then we will obtain $u(t)$, which is the input of the spherical mobile robot, thus, we finally get the near-optimal trajectory of the robot.

V. SIMULATION AND EXPERIMENT OF MOTION PLANNING

A. Simulation

Assuming spherical robot BHQ-2 rolls without slipping on a plane, and suppose its initial position and pose to be $P = [x_0, y_0, \psi_0, \beta_0]^T = [0,0,0,0]^T$, and its final position and pose to be $p_g = [x_g \;\; y_g \;\; \psi_g \;\; \beta_g]^T = [1 \;\; 1 \;\; 0 \;\; 0]^T$. In order to simplify the calculation, we set the motion time of the spherical robot to be $T = 6.28s$, and choose the former six Fourier basis as the orthonormal basis of $L_2([0,T])$, namely

$$e_1 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}, \; e_2 = \begin{bmatrix} \sin t \\ 0 \end{bmatrix}, \; e_3 = \begin{bmatrix} \cos t \\ 0 \end{bmatrix},$$

$$e_4 = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \; e_5 = \begin{bmatrix} 0 \\ \sin t \end{bmatrix}, \; e_6 = \begin{bmatrix} 0 \\ \cos t \end{bmatrix}$$

Set $\gamma = 100$, then we simulate the iterative computing with software Matlab, and after 28 times of iterative computing, we have

$$\alpha = [5.0611, -0.0520, 0.6290, \\ -0.0054, 0.0108, 0.2679]^T$$

Therefore the input of the spherical robot is

$$u = \begin{bmatrix} \dot{\varphi} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 5.0611 \times 0.5 - 0.0520 \times \sin t - 0.6290 \times \cos t \\ -0.0054 \times 0.5 + 0.0108 \times \sin t + 0.5679 \times \cos t \end{bmatrix}$$

We plot the simulating optimal motion trajectory and the input curve in Fig.2 and Fig.3.

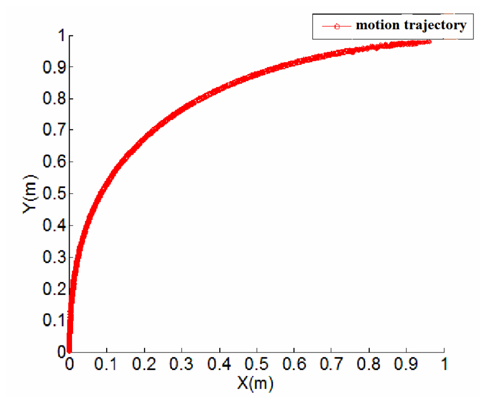

Figure 2. Simulated optimal trajectory

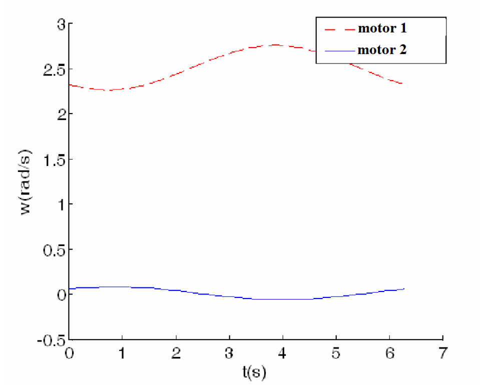

Figure 3. The input curve

The curves of pitch angle and the roll angle of the robot are show in Fig.4 and Fig.5.

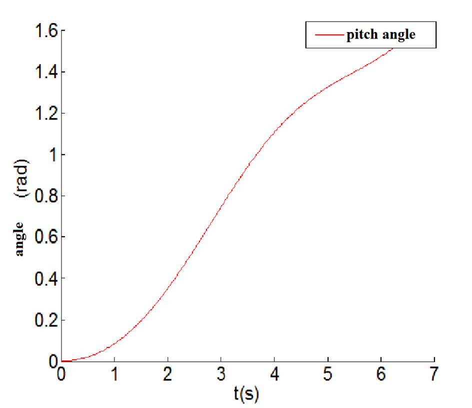

Figure 4. The pitch angle curve of the robot

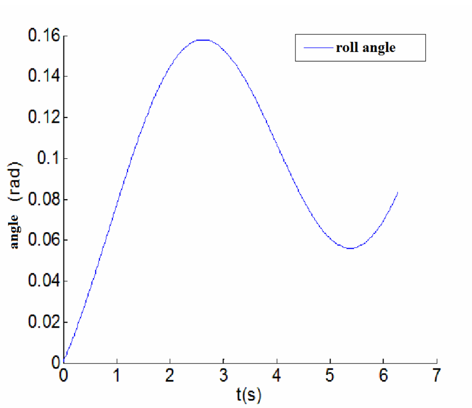

Figure 5. The roll angle curve of the robot

From the simulation we can find that the method discussed above can realize the optimal trajectory planning for the spherical mobile robot, since we have ignored the higher order polynomials during the Taylor series expansion, the planning results have some small errors.

B. Experiment

We process the iterative computing of the trajectory planning algorithm discussed above by a PC to obtain the input $u(t)$, and then sending it to BHQ-2 by the wireless link in real time manner, and then the DSP controller inside BHQ-2 to realize the planning by control the two motors with PID method. A vision system to observe the trajectory of BHQ-2 was also developed, and which is made up of a JVC TK-C1381 color camera, a Daheng CG-300 frame grabber, a tripod to carry the camera, and an IBM laptop, the experiment environment is a corridor of a building, where the ground is flat. The setup of the experimental system is shown in Fig.6.

Figure 6. The experiment system

In order to get the trajectory of BHQ-2 from the frames gathered during its moving, images are processed as following: firstly, filter the noises by a mean filter, then apply the frame difference method to obtain the exact area of BHQ-2 in each image, after that the Canny edge operator and mathematical morphology operations are used to produce the binary images that express the exact edge of BHQ-2 to be a set of circles, and in the final step, we apply the Hough transformation to detect the centroid coordinate of the spherical robot in the image. The basic idea of Hough transformation is to transform the image space into the parameter space, and then curves in image are parametrized in a certain form that most edge points fit, by accumulating and filtrating, we can obtain the preferred information in images. By calculating series of images of the motion of spherical mobile robot BHQ-2, we can get its motion trajectory. In one experiment, one image after filtering is shown in Fig.7 and the detected trajectory of BHQ-2 moving with the trajectory planning method is shown in Fig.8.

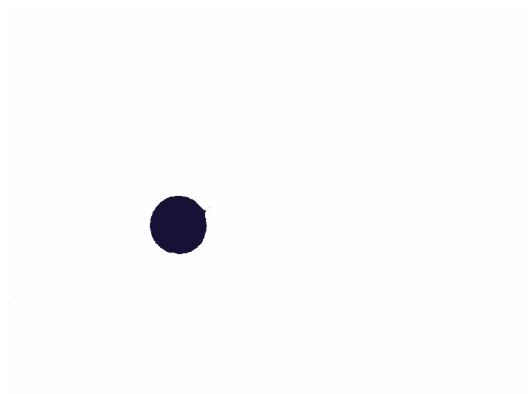

Figure 7. Filtered image of BHQ-2

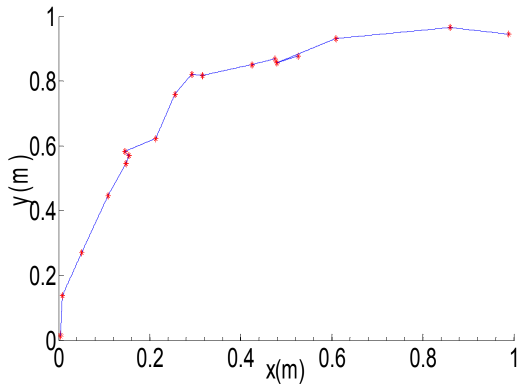

Figure 8. The experimental trajectory of BHQ-2

The results show that with the motion planning method we have discussed, BHQ-2 not only can move from the initial point to the desired point with the set pose, but also follow a path almost the same as the trajectory we obtained from simulation (comparing Fig.2.) with the same parameters. In Fig.8, there are some small errors and shock in the trajectory, which are mainly caused due to the internal mechanical structure error of BHQ-2 and the open-loop motor control method. Several experimental results show that the real motion trajectories generally meet the simulation ones and the average errors is about 8%. So the discussed trajectory planning method for spherical mobile robot BHQ-2 is effective and feasible.

VI. Conclusion

The paper briefly introduces a spherical mobile robot BHQ-2 designed for environment exploration and mainly discusses its control modeling and near-optimal nonholonomic motion planning. By analyzing the kinematic equations of the BHQ-2 system, we knew that this is a nonholonomic constrain system. Then the controllability of this system was proved by using the Lie algebra. Based on the kinematic equation of BHQ-2, the trajectory planning of BHQ-2 was researched with the Gauss-Newton algorithm. Simulation and experimental results show that effectively resolve the nonholonomic motion planning problem of spherical mobile robot BHQ-2. The method can realize the motion planning and the optimization of the control rules as well as fast convergence speed. However, due to ignoring the high order items in Taylor expansion, the motion planning results have some errors. In real application, the motion accuracy can be improved by selecting suitable Fourier orthogonal basis vectors and parameter γ. From experiments we also find that the motion trajectory of the robot can be greatly affected by external interference under open-loop control. It was mainly because that the actual trajectory could not be detected and no active stabilization strategy was imposed on the robot. So closed-loop control methods including dynamics need to be researched in the future in order to get high precision motion.

Acknowledgment

This work is supported in part by National Natural Science Foundation of China under the grant No. 50705003 and National High Technology Research and Development Program of China under the grant No. 2007AA04Z252.

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