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A novel Max-Plus algebra based wavelet transform and its applications in Image Processing

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***Abstract*—Max-plus algebra based wavelet transforms are considered currently to be the world's fastest image processing methods. In the present paper a novel max-plus algebra based wavelet transform is proposed and studied both from the theoretical and practical points of view. The novel MP-wavelets considered here employ strictly neighbor pixels in the calculations and their most important properties are: very low computational complexity, flexible sampling window size and potentially, very easy hardware implementation.**

***Index Terms*—max-plus algebra, MP-wavelets, mathematical morphology, image coding**

I. Introduction

Nowadays, research efforts of several researchers in image processing are directed towards new image and video coding methods that satisfy the following properties:

- They should keep reasonable quality of images or video;
- They should have low computational complexity;
- They should be suitable for hardware implementation;
- They should be suitable for copyright protection;
- They should be adaptive to the application at hand.

The problem of designing cheap methods for the surveillance and storage of data collected from surveillance applications also motivates a search for new innovative methods.

A nonlinear wavelet-type transform was recently proposed in Heijmans' works [4] [5], i. e., morphological wavelets, however, the morphological wavelets are defined based on the ordinal algebra on real numbers $\mathbb{R}$ together with the four arithmetic operations supplemented by max and min.

Max-plus (Tropical) algebra ([1], [9], [3], [11]) is an interesting algebraic structure (considered exotic by several Mathematicians [6]) over the set of integers $\mathbb{Z}$ or reals $\mathbb{R}$, endowed with maximum, minimum and standard addition as operations. It is interesting to remark that the idea to use max-plus algebra has recently appeared in many fields of mathematics. Tropical Algebra and Tropical Geometry ([10]) are fields of an increasing interest in the Mathematics Community. Also, Mathematical Morphology ([5]) is a novel, fruitful idea in the area of Computer Science.

Definition. Let us consider the (extended) set of integers $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{-\infty, \infty\}$, where $\mathbb{Z}$ is the set of integers. Over the set $\bar{\mathbb{Z}}$ one considers the operations $\vee$, $\wedge$ and $+$ ($a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$). The algebraic structure

$$\mathcal{Z}_{max} = (\bar{\mathbb{Z}}, \vee, \wedge, +, -\infty, \infty, 0) \quad (1)$$

is called *max-plus algebra,* and the following properties hold true:

- $\vee$ and $\wedge$ are associative and commutative with neutral elements $-\infty$ and ∞ respectively;
- $(\mathbb{Z}, +, 0)$ is the standard additive group of integers;
- $+$ is distributive with respect to $\vee$ and $\wedge$;
- $\vee$ is distributive with respect to $\wedge$ and viceversa;
- $-\infty$ is an absorbing element for $\wedge$ while ∞ is absorbing element for $\vee$.

The convention that $\wedge$ and $\vee$ have a higher priority than $+$ is used in the present paper.

Max-Plus algebra based wavelet transforms are defined using a multiresolution analysis approach. They are built using a nonlinear Analysis and a nonlinear Synthesis operator. The analysis operator is composed of an "approximation" mapping $\varphi_k^{\uparrow} : I_k \to I_{k+1}$ and a "details" mapping $\omega_k^{\uparrow} : I_k \to D_{k+1}$, while the synthesis operator is of the form $\psi_k^{\downarrow} : I_{k+1} \times D_{k+1} \to I_k$, where I_k and D_k, $k \geq 0$ are nonempty. For a wavelet decomposition scheme the fulfillment of the pyramid condition (perfect reconstruction property) is essential. The pyramid condition reads $\psi_k^{\downarrow} \circ (\varphi_k^{\uparrow}, \omega_k^{\uparrow}) = 1_{I_k}$, where $\circ$ denotes the usual composition and 1_{I_k} is the identity mapping over I_k. The analysis process of MP-Wavelets using two types of signal decompositions: 'approximation' ($\varphi_k^{\uparrow}$ in our case) to generate approximation signals in the space I_{k+1} and 'detailed' ($\omega_k^{\uparrow}$) to generate detailed signals in the space D_{k+1}. The synthesis operator $\psi_k^{\downarrow}$ performs the reconstruction.

In Fig. 1 an example of a wavelet decomposition scheme is shown, using a sampling window size of 4x4 pixels. One approximation and three detail components are obtained.

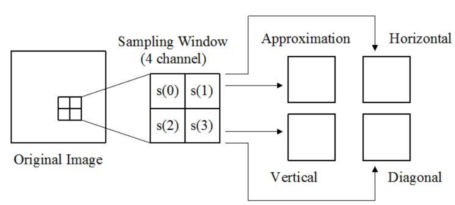

Fig. 1. A max-plus wavelet decomposition scheme with 4x4 sampling window size

In the present paper the operators $\varphi_k^{\uparrow}$, $\omega_k^{\uparrow}$ and consequently $\psi_k^{\downarrow}$ are all nonlinear operators. Also the spaces I_k and D_k, $k \geq 0$ are considered with a max-plus structure as considered above.

As key for the perfect reconstruction property, we will employ the following two identities:

$$a \wedge b_1 \wedge b_2 \wedge ... \wedge b_n + (a-b_1) \vee (a-b_2) \vee ... \vee (a-b_n) \vee 0 = a, \quad (2)$$

and the dual

$$a \vee b_1 \vee b_2 \vee ... \vee b_n + (a-b_1) \wedge (a-b_2) \wedge ... \wedge (a-b_n) \wedge 0 = a, \quad (3)$$

for any $a, b_1, b_2, ..., b_n \in \mathbb{Z}$.

It is easy to check the validity of these identities. Indeed, if e.g.

$$a \wedge b_1 \wedge ... \wedge b_n = b_k$$

then

$$(a - b_1) \vee ... \vee (a - b_n) = a - b_k > 0$$

and so

$$a \wedge b_1 \wedge ... \wedge b_n + (a-b_1) \vee ... \vee (a-b_n) \vee 0 = b_k + (a-b_k) = a.$$

On the other hand if

$$a \wedge b_1 \wedge ... \wedge b_n = a$$

then

$$(a - b_1) \vee ... \vee (a - b_n) \leq 0$$

thus

$$a \wedge b_1 \wedge ... \wedge b_n + (a - b_1) \vee ... \vee (a - b_n) \vee 0 = a.$$

II. Type I MP-wavelets

Type I and II MP wavelets were proposed and studied recently in [8], [7]. Let us recall here the definition of the Type I MP-wavelets.

The analysis operator is given by its approximation component $\varphi_k^{\uparrow} : I_k \rightarrow I_{k+1}$, and its detail components $\delta_{k,i}^{\uparrow} : I_k \rightarrow D_{k+1,i}$, $i = 1, ..., p-1$

$$\varphi_k^{\uparrow}(x)_n = y_{pn} = x_{pn} \wedge x_{pn+1} \wedge ... \wedge x_{pn+p-1}$$
$$\delta_{k,i}^{\uparrow}(x)_n = y_{pn+i} = x_{pn} - x_{pn+i}.$$

In this case the synthesis is $\psi_k^{\downarrow} : I_{k+1} \times D_{k+1,1} \times ... \times D_{k+1,p-1} \rightarrow I_k$,

$$\psi_k^{\downarrow}(y)_{pn} = z_{pn} = y_{pn} + y_{pn+1} \vee ... \vee y_{pn+p-1} \vee 0$$
$$\psi_k^{\downarrow}(y)_{pn+i} = z_{pn+i} = z_{pn} - y_{pn+i}.$$

It is easy to check that the pyramid condition holds by (2).

III. Type II MP-wavelets

The application of the type I, one-dimensional wavelet decomposition scheme successively on the horizontal and vertical direction gives a new type of MP-wavelets, different from type I MP-wavelets introduced above. The scheme is a biorthogonal-like wavelet decomposition scheme. The analysis is performed using a one dimensional type I scheme first on the horizontal, followed by the vertical direction. The synthesis is performed successively applying the 1-D synthesis scheme on the vertical direction followed by the 1-D scheme applied on the horizontal direction. As it was shown in [7] the perfect reconstruction property holds for this type of MP-wavelet too.

IV. Type III MP-wavelets

Let us first observe that Type I MP-wavelets have the detail component of the analysis operator defined based on a preferred center pixel x_{pn} ($\delta_{k,i}^{\uparrow}(x)_n = x_{pn} - x_{pn+i}$). The idea in the definition of MP-wavelets of type III is to use identity (2) in such a way that the neighbor pixels are involved in operations instead of a preferred pixel.

The analysis operator is given by its approximation component $\varphi_k^{\uparrow} : I_k \rightarrow I_{k+1}$, and the details $\omega_k^{\uparrow}$ having possibly several components $\omega_k^{\uparrow} = \left(\delta_{k,i}^{\uparrow}\right)_{i=1,...,p-1}$, where $\delta_{k,i}^{\uparrow} : I_k \rightarrow D_{k+1,i}$, for any $i = 1, ..., p-1$

$$\varphi_k^{\uparrow}(x)_n = y_{pn} = x_{pn} \wedge x_{pn+1} \wedge ... \wedge x_{pn+p-1}$$
$$\delta_{k,i}^{\uparrow}(x)_n = y_{pn+i} = x_{pn+i-1} - x_{pn+i}.$$

The synthesis in this case is $\psi_k^{\downarrow} : I_{k+1} \times D_{k+1,1} \times ... \times D_{k+1,p-1} \rightarrow I_k$,

$$\psi_k^{\downarrow}(y)_{pn} = z_{pn} = y_{pn} + 0 \vee y_{pn+1} \vee (y_{pn+1} + y_{pn+2}) \vee ... \vee (y_{pn+1} + ... + y_{pn+p-1})$$
$$\psi_k^{\downarrow}(y)_{pn+i} = z_{pn+i} = z_{pn+i-1} - y_{pn+i}.$$

It is easy to check that the pyramid condition holds by (2). Indeed, using the notations introduced as above,

$$\psi_k^{\downarrow}(\varphi_k^{\uparrow}(x), \delta_{k,i}^{\uparrow}(x))_{pn} = y_{pn} + 0 \vee y_{pn+1} \vee (y_{pn+1} + y_{pn+2}) \vee ... \vee (y_{pn+1} + ... + y_{pn+p-1})$$
$$= x_{pn} \wedge x_{pn+1} \wedge ... \wedge x_{pn+p-1}$$
$$+ 0 \vee (x_{pn} - x_{pn+1}) \vee (x_{pn} - x_{pn+2}) \vee (x_{pn} - x_{pn+p-1})$$
$$= x_{pn}$$

and by induction, if we suppose that $z_{pn+i-1} = x_{pn+i-1}$, then

$$\psi_k^{\downarrow}(\varphi_k^{\uparrow}(x), \delta_{k,i}^{\uparrow}(x))_{pn+i} = z_{pn+i-1} - y_{pn+i}$$
$$= x_{pn+i-1} - (x_{pn+i-1} - x_{pn+i}) = x_{pn+i},$$

so, $\psi_k^{\downarrow} \circ (\varphi_k^{\uparrow}, \omega_k^{\uparrow}) = 1_{I_k}$.

Since $\delta_{k,i}^{\uparrow}(x)_n = x_{pn+i-1} - x_{pn+i}$ is a difference of colors of neighbor pixels one can expect that these MP-wavelets will work well with higher sampling window sizes as compared to MP-wavelets of types I.

We consider in the followings a biorthogonal-like transform for two-dimensional signals as follows. A Type-III MP-wavelet

Fig. 2. Original Image (Lenna)

Fig. 3. Reconstructed Images (3 × 1 channel), Left: Comp. Rate = 0.306, RMSE = 20.53, Right: Comp. Rate = 0.273, RMSE = 30.07

analysis on the horizontal direction is followed by a Type III MP-wavelet analysis on the vertical direction. In order to reconstruct the original signal we employ a Type III MP-wavelet synthesis on the vertical direction followed by the synthesis on the horizontal direction.

V. Experimental results

An image compression/reconstruction experiment is performed, in order to study the properties of the new MP-wavelets and also to compare MP-wavelets of different types, with various sizes of the sampling windows. In this experiment, we use 3×1, 5×1, and 5×5 windows in Type-III MP-wavelets. The original image is shown in Fig. 2. It is easy to check also, that for the window size 3×3, the MP-wavelet of type I is the same as the MP-wavelet of type III. MP-wavelets of type II are not showing a significant difference with respect to type I MP-wavelets (see [7]) so we use the later ones for comparison in the present section.

The reconstructed images are shown in Figs. 3, 4, and 5, respectively.

The RMSE (Root Mean Square Error) comparison of 3×1, 5×1, and 5×5 windows in Type-III MP-wavelets is also presented in Fig. 6. As can be seen from the comparison, the performance of 5×5 window is better than that of 3×1, and 5×1.

Also, if we compare Type I and Type III MP-wavelets, we can conclude the followings. If the sampling window size is 3×1, 5×1, then there is not a significant difference in the performances of type I and III MP-wavelets. If the sampling window size is 3×3, then the two wavelets (type I and III)

Fig. 4. Reconstructed Images (5 × 1 channel), Left: Comp. Rate = 0.195, RMSE = 20.50, Right: Comp. Rate = 0.171, RMSE = 30.25

Fig. 5. Reconstructed Images (5 × 5 channel), Left: Comp. Rate = 0.034, RMSE = 19.96, Right: Comp. Rate = 0.029, RMSE = 28.54

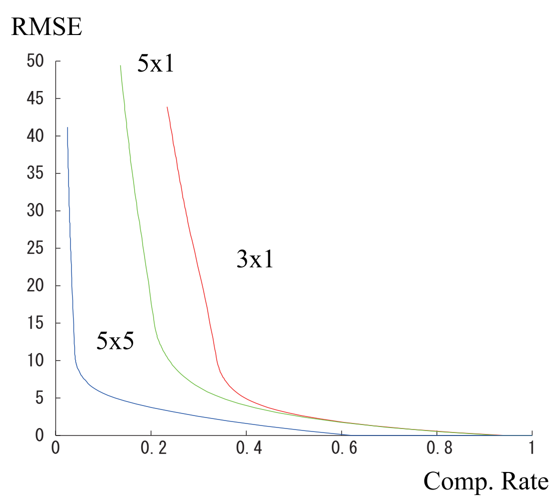

Fig. 6. RMSE Comparison for Type III MP-wavelet with different sampling window sizes

coincide. If a sampling window size of 5×5 pixels is used, type III MP-wavelets outperform considerably type I MP-wavelets. Indeed, in Figure 8 reconstruction results are presented for image Lenna using Type I and Type III MP-wavelets. Comparison between performances of 5x5 window based Type I and Type III MP-wavelets on different compression rates is analyzed in Fig. 7. The conclusion is that MP-wavelets of type III outperform their type I counterparts. This shows that higher sampling window size can be used with Type III MP-wavelets.

A. *Conlusions*

The present paper introduces and studies a new type of MP-wavelet based image coding method that keeps a good quality together with keeping the method very simple. It also works well with different window sizes. Future research is directed towards video coding using type III MP-wavelets.

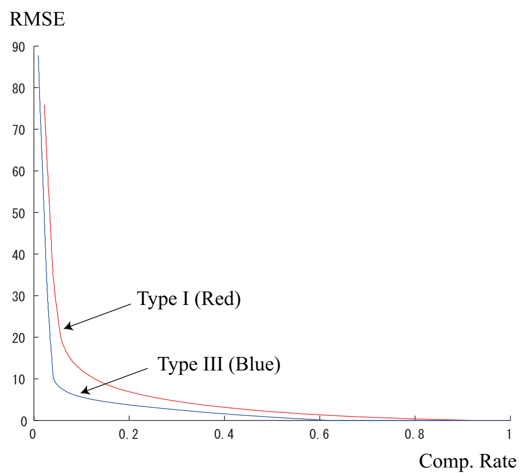

Fig. 7. RMSE Comparison of type I and type III MP-wavelets with 5x5 sampling window size

Fig. 8. Reconstructed Images (5×5 channel), Left: Type I, Comp. Rate = 0.098, RMSE = 12.26, Right: Type III, Comp. Rate = 0.099, RMSE = 5.65

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