Induced Aggregation Operators in the Ordered Weighted Average Sum

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***Abstract*—The ordered weighted average (OWA) aggregation is an extension of the classical weighted average by using a reordering process of the arguments in a decreasing or increasing way. This article presents new averaging aggregation operators by using sums and order inducing variables. This approach produces the induced ordered weighted average sum (IOWAS). The IOWAS operator aggregates a set of sums using a complex reordering process based on order-inducing variables. This approach includes a different types of aggregation structures including the well-known OWA families. The work presents additional generalizations by using generalized and quasi-arithmetic means. The paper ends with a simple numerical example that shows how to aggregate with this new approach.**

Keywords—ordered weighted average; induced aggregation operators; sums; quasi-arithmetic means

I. INTRODUCTION

Averaging aggregation operators are very common in the literature [1-2]. They summarize the information of a set of data. A very well-known averaging aggregation operator is the ordered weighted average (OWA) operator [3-4]. It is an averaging aggregation operator that collects the data of a set providing a parameterized family of aggregation operators between the minimum and the maximum. The OWA operator orders the data in a decreasing or increasing way. However, in many problems it is necessary to consider more complex reordering processes of the information. An alternative operator to the OWA operator that is able to do so is the induced OWA operator [5]. It aggregates the information similarly to the OWA operator but with an initial reordering process that can take any form according to order-inducing variables. Recently, the IOWA operator has received a lot of attention by many authors. Some authors suggested geometric versions [6-7]. Merigó and Gil-Lafuente [8] used generalized and quasi-arithmetic means in the aggregation. Other authors extended this approach to uncertain environments where the information can be assessed with interval numbers [9], fuzzy numbers [10-11], intuitionistic fuzzy sets [12-13] and linguistic variables [14]. Some other studies have developed other extensions by using other types of aggregations including weighted averages [15], probabilities [16], Choquet integrals [17-18], moving averages [19] and distance measures [20].

Recently, Merigó and Yager [21] introduced the ordered weighted average sum (OWAS). It is an aggregation operator that aggregates a set of sums from the lowest sum to the highest one. By doing so, this approach can analyze a set of sums considering the attitudinal character of the decision maker. They also presented several generalizations by using generalized and quasi-arithmetic means and Choquet integrals. Note that the OWAS operator can be seen as a particular case of OWA norms (OWAN) [22].

The aim of this paper is to present a more general framework of the OWAS operator by using induced aggregation operators. The article introduces the induced ordered weighted average sum (IOWAS). This averaging aggregation operator deals with a set of sums summarizing the information through the use of order-inducing variables that represent complex attitudes in the analysis of the data. The work studies some key properties and some families of IOWAS operators including the average sum, the step-IOWAS and the olympic-IOWAS operator.

The article also presents further generalizations of the IOWAS operator by using generalized and quasi-arithmetic means obtaining the induced generalized OWA sum (IGOWAS) and the induced Quasi-OWA sum (Quasi-IOWAS). These operators include a wide range of particular cases including the geometric induced OWA sum (GIOWAS), the induced quadratic OWA sum (IQOWAS), and the induced harmonic OWA sum (IHOWAS). The study ends analyzing the applicability of the new approach in different fields and developing a simple numerical example of an aggregation process of sums in the analysis of variable and fixed costs in an enterprise.

The rest of the paper is organized as follows. Section 2 briefly reviews the preliminaries of the paper. Section 3 introduces the IOWAS operator and some of its key properties. Section 4 presents the Quasi-IOWAS operator and some representative particular cases. Section 5 develops an illustrative example and Section 6 summarizes the main conclusions and findings of the article.

II. PRELIMINARIES

A. The Ordered Weighted Average

The OWA operator [3] is an averaging aggregation operator that has received a lot of attention during the last years [4]. OWA aggregates the data giving a parameterized system of aggregation operators that move between the maximum and the minimum. The OWA operator is formulated as follows.

Definition 1. An OWA operator of dimension n is a mapping OWA: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$OWA\,(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j\,, \tag{1}$$

where b_j is the jth largest of the a_i.

B. Induced Aggregation Operators

The IOWA operator was introduced by Yager and Filev [5] and its main difference with the OWA operator is that the reordering step is not developed with the values of the arguments a_i. In this case, the reordering step is developed with order-inducing variables. It can be defined as follows.

Definition 2. An IOWA operator of dimension n is a mapping $IOWA$: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$IOWA\,(\langle u_1, a_1\rangle, \langle u_2, a_2\rangle, \ldots, \langle u_n, a_n\rangle) = \sum_{j=1}^{n} w_j b_j\,, \tag{2}$$

where b_j is the a_i value of the IOWA pair $\langle u_i, a_i\rangle$ having the jth largest u_i, u_i is the order-inducing variable and a_i is the argument variable.

The OWA operator is included as a particular case when the ordering provided by the order inducing variables is equivalent to a numerical decreasing order of the arguments. The IOWA operator can be generalized with generalized and quasi-arithmetic means. The result is the IGOWA and the Quasi-IOWA operator. The Quasi-IOWA operator can be defined as follows.

Definition 3. A Quasi-IOWA operator is a mapping $QIOWA$: $R^n \times R^n \rightarrow R$ with a weighting vector W of dimension n, $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$QIOWA\,(\langle u_1, a_1\rangle, \ldots, \langle u_n, a_n\rangle) = g^{-1}\left(\sum_{j=1}^{n} w_j g(b_j)\right), \tag{3}$$

where $(b_1, \ldots, b_n)$ is simply $(a_1, \ldots, a_n)$ reordered in decreasing order of the values of the u_i, u_i is the order-inducing variable, a_i is the argument variable and g is a strictly continuous monotonic function.

Note that the Quasi-IOWA operator includes a wide range of aggregation operators. For further reading on the IOWA, refer, e.g., to Merigó and Gil-Lafuente [8] and Yager et al. [4].

C. The Ordered Weighted Average Sum

The ordered weighted average sum (OWAS) is an aggregation operator that aggregates a set of sums from the minimum sum to the maximum one. It is very useful for dealing with the aggregation of sums which is very common in business and economics when dealing with the sum of economic variables such as the costs, sales, benefits and assets. The OWAS operator can be defined as follows for two sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$.

Definition 4. An OWAS operator of dimension n is a mapping $OWAS$: $R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$OWAS\,([x_1 + y_1], \ldots, [x_n + y_n]) = \sum_{j=1}^{n} w_j b_j\,, \tag{4}$$

where b_j is the jth largest of the $[x_i + y_i]$.

The OWAS operator accomplishes other properties similar to the common OWA operators [3-4]. Moreover, it includes a wide range of particular cases including the average sum, the step-OWAS and the median sum [21,24].

The attitudinal character of the OWAS operator can also be measured through the degree of orness measure [3] as follows:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right). \tag{5}$$

III. THE INDUCED ORDERED WEIGHTED AVERAGE SUM

A. Theoretical Foundations

The induced OWA sum (IOWAS) is an averaging aggregation operator that aggregates a set of sums from the minimum to the maximum and by using a complex reordering process that is carried out with order-inducing variables. The main advantage of this approach is that it represents a general framework for aggregating sums between the minimum and the maximum and adapting the results to the specific interests of the decision maker. The IOWAS operator is defined as follows.

Definition 5. An IOWAS operator of dimension n is a mapping $IOWAS$: $R^n \times R^n \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$IOWAS\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j\,, \tag{6}$$

where b_j is the $[x_i + y_i]$ value of the IOWA triplet $\langle u_i, x_i, y_i\rangle$ having the jth largest u_i, u_i is the order-inducing variable and $[x_i + y_i]$ is the argument variable represented in the form of a sum.

Observe that the IOWAS operator is symmetric. Therefore, although Definition 5 reorders the order-inducing variables in a decreasing way, it is also possible to consider an ascending IOWAS (AIOWAS) operator by using $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the descending IOWAS operator and w^*_{n-j+1} the jth weight of the AIOWAS operator.

When $W = \sum_{j=1}^{n} w_j \neq 1$, then, the IOWAS operator can be formulated in a more general way as:

$$IOWAS\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \frac{1}{W}\sum_{j=1}^{n} w_j b_j \,. \tag{7}$$

The IOWAS operator is commutative, monotonic, idempotent and bounded. This can be proved with the following theorems.

Theorem 1 (Commutativity of IOWA aggregation). *Suppose f is the IOWAS operator, then*

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = f\,(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle), \tag{8}$$

where $(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle)$ *is any permutation of the arguments* $(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle)$.

Proof. Let

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j \,, \tag{9}$$

$$f\,(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle) = \sum_{j=1}^{n} w_j e_j \,. \tag{10}$$

Because $(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle)$ is a permutation of $(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle)$, and we have $|x_i + y_i| = |c_i + d_i|$, for all i, and then

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = f\,(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle). \;\blacksquare$$

Note that the commutativity of the IOWAS can also be studied from the context of a distance measure, which can be proved with the following theorem.

Theorem 2 (Commutativity – sum). *Suppose f is the IOWAS operator, then*

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = f\,(\langle u_1, y_1, x_1\rangle, \ldots, \langle u_n, y_n, x_n\rangle). \tag{11}$$

Proof. Let

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j \,, \tag{12}$$

$$f\,(\langle u_1, y_1, x_1\rangle, \ldots, \langle u_n, y_n, x_n\rangle) = \sum_{j=1}^{n} w_j e_j \,. \tag{13}$$

Because $|x_i + y_i| = |y_i + x_i|$, for all i, then

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = f\,(\langle u_1, y_1, x_1\rangle, \ldots, \langle u_n, y_n, x_n\rangle). \;\blacksquare$$

Theorem 3 (Monotonicity). *Suppose f is the IOWAS operator; if* $|x_i + y_i| \geq |c_i + d_i|$, *for all* i_i, *then*

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \geq f\,(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle). \tag{14}$$

Proof. Let

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j \,, \tag{15}$$

$$f(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle) = \sum_{j=1}^{n} w_j e_j \,. \tag{16}$$

Because $|x_i + y_i| \geq |c_i + d_i|$, for all i, then

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \geq f\,(\langle u_1, c_1, d_1\rangle, \ldots, \langle u_n, c_n, d_n\rangle). \;\blacksquare$$

Theorem 4 (Boundary condition). *Suppose f be the IOWAS operator, then*

$$\min\{|x_i + y_i|\} \leq f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \leq \max\{|x_i + y_i|\}. \tag{17}$$

Proof. Let $\max\{|x_i + y_i|\} = c$ and $\min\{|x_i + y_i|\} = d$; then

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j \leq \sum_{j=1}^{n} w_j c = c\sum_{j=1}^{n} w_j \,, \tag{18}$$

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j \geq \sum_{j=1}^{n} w_j d = d\sum_{j=1}^{n} w_j \,. \tag{19}$$

Because $\sum_{j=1}^{n} w_j = 1$, we get

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \leq c, \tag{20}$$

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \geq d. \tag{21}$$

Therefore,

$$\min\{|x_i + y_i|\} \leq f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) \leq \max\{|x_i + y_i|\}. \blacksquare$$

Theorem 5 (Idempotency). *Let f be the IOWAS operator; if* $|x_i + y_i| = a$, *for all i, then*

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = a. \tag{22}$$

Proof. Because $|x_i + y_i| = a$, for all i, we have

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{j=1}^{n} w_j b_j = \sum_{j=1}^{n} w_j a = = a\sum_{j=1}^{n} w_j \,. \tag{23}$$

Because $\sum_{j=1}^{n} w_j = 1$, we get

$$f\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = a. \qquad \blacksquare$$

The weighting vector of the IOWAS operator can be characterized by using a wide range of measures [3,8]. The degree of orness, also known as the degree of optimism, is similar to the degree of orness of the OWA operator although it has some important differences [3]. It is formulated as follows:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right). \tag{24}$$

Observe that here the orness assumes that the ordering given by the order-inducing variables gives first the maximum and so on. Sometimes it is possible to assume this when the numerical information is not linearly ordered in a decreasing way. An example of this could be the temperature of the human body. However, in order to analyse the orness in a similar way to the OWA operator [3], we should reorder the weights according to the numerical values of the arguments in a decreasing way.

The entropy of dispersion of the IOWAS weighting vector uses the same formulation as in the OWA operator [3] and has strong similarities with the Shannon entropy [23]. It is defined as:

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j). \tag{25}$$

Note that the maximum entropy appears with the arithmetic mean while the lowest one occurs when all the weight is given to one weight as in the step-OWA operator [24].

The balance operator of the IOWAS operator uses the same structure as in the OWA operator and its generalizations [8,25]:

$$BAL\,(W) = \sum_{j=1}^{n}\left(\frac{n+1-2j}{n-1}\right)w_j\ . \tag{26}$$

Note that the assumption is that Eq. (26) measures the balance of the weights. However, from a numerical perspective, the balance to the maximum and the minimum should be measured considering that the first weight should aggregate the maximum, the second weight the second largest and so on.

Finally, it is worth noting that all the analysis presented for the IOWAS operator has been developed for two sets. However, it is possible to extend this approach to m sets. In this case, we could extend Eq. (6) in the following way.

Definition 6. An IOWAS operator of dimension n is a mapping $IOWAS$: $R^n \times R^n \times \ldots \times R^n \rightarrow R$ that has an associated weighting vector W of dimension n with $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$f(\{u_1, x_1^1, \ldots, x_1^m\}, \ldots \{u_n, x_n^1, \ldots, x_n^m\}) = \sum_{j=1}^{n} w_j b_j\ , \tag{27}$$

where b_j is the $[x_i^1 + \ldots + x_i^m]$ value of the IOWA set $\langle u_i, x_i^1, \ldots, x_i^m\rangle$ having the jth largest u_i, u_i is the order-inducing variable and $[x_i^1 + \ldots + x_i^m]$ is the argument variable given in the form of a sum.

B. Families of IOWAS Operators

The IOWAS operator includes many different types of aggregation operators by using different expressions in the weighting vector. Among others, let us look into the following ones:

The maximum sum is found if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \mathrm{Max}\{a_i\}$).

The minimum sum appears if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and $u_p = \mathrm{Min}\{a_i\}$.

More generally, if $w_k = 1$ and $w_j = 0$ for all $j \neq k$, we get the step-IOWAS operator.

If $w_j = 1/n$ for all i, the IOWAS operator is converted into the simple average sum (AS) as follows:

$$AS\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \frac{1}{n}\sum_{i=1}^{n}(x_i + y_i)\,, \tag{28}$$

Although it is not strictly a particular case of the OWAS operator, it is also interesting to mention the weighted average sum which is defined as follows:

$$WAS\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \sum_{i=1}^{n} w_i(x_i + y_i)\,, \tag{29}$$

The OWAS operator (Eq.(4)) appears if the ordered position of the order-inducing variables u_i of the IOWAS operator have the same ordering as the decreasing order generated by the numerical values of $|x_i + y_i|$, for all i.

The olympic-IOWAS operator is formed when $w_1 = w_n = 0$, and for all others, $w_{j^*} = 1/(n-2)$. That is:

$$f(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \frac{1}{n-2}\sum_{j=2}^{n-1} b_j\ , \tag{30}$$

Note that Eq. (20) could be generalized with $w_j = 0$ for $j = 1, 2, \ldots, k, n, n-1, \ldots, n-k+1$; and for all others, $w_{j^*} = 1/(n-2k)$, where $k < n/2$. This formulation would represent a generalized olympic-IOWAS operator.

Note that a wide range of other expressions in the weighting vector of the IOWAS operator could be studied following the OWA and IOWA literature [8,24,26-27].

IV. Generalized Means in the IOWAS Operator

The IOWAS operator is an arithmetic averaging aggregation operator. However, it is possible to extend it by utilizing generalized and quasi-arithmetic means [8,28-29]. By using generalized means, we obtain the induced generalized OWAS (IGOWAS) operator. It is defined as follows for two sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$:

Definition 7. An IGOWAS operator of dimension n is a mapping $IGOWAS$: $R^n \times R^n \times R^n \rightarrow R$ with a weighting vector W of dimension n where $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, such that:

$$IGOWAS\,(\langle u_1, x_1, y_1\rangle, \ldots, \langle u_n, x_n, y_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda}\ , \tag{31}$$

where b_j is the $[x_i + y_i]$ value of the IOWA triplet $\langle u_i, x_i, y_i\rangle$ having the jth largest u_i, u_i is the order-inducing variable, $[x_i +$

y_i] is the argument variable and λ is a parameter such that $\lambda \in \{-\infty, \infty\} - \{0\}$.

Note that the IGOWAS operator can be further generalized obtaining the induced quasi-arithmetic OWAS (Quasi-IOWAS) operator. It is formulated as follows:

$$QIOWAS(\langle u_1, x_1, y_1\rangle,\ldots,\langle u_n, x_n, y_n\rangle) = g^{-1}\left(\sum_{j=1}^{n} w_j g(b_j)\right), \quad (32)$$

where g is a strictly continuous monotonic function.

The main advantage of the IGOWAS and the Quasi-IOWAS operator is that they provide a more general formulation that includes a wide range of particular aggregation operators including quadratic, cubic, harmonic and geometric aggregations.

The IGOWAS operator can also consider more than two sets as it is shown in Eq. (27) for the IOWAS operator. In this case, the formula would be as follows:

$$f(\{u_1, x_1^1,\ldots,x_1^m\},\ldots\{u_n, x_n^1,\ldots,x_n^m\}) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda}. \quad (33)$$

The IGOWAS operator includes a wide range of particular types of aggregation by using a different expression in the parameter λ. For example, for m sets:

- If $\lambda = 1$, the IGOWAS is converted into the usual IOWAS operator.
- If $\lambda = 2$, we obtain the quadratic IOWAS (IOWQAS) operator.

$$f(\{u_1, x_1^1,\ldots,x_1^m\},\ldots\{u_n, x_n^1,\ldots,x_n^m\}) = \sqrt{\left(\sum_{j=1}^{n} w_j b_j^2\right)}. \quad (34)$$

- If $\lambda = 3$, we obtain the cubic IOWAS (IOWCAS) operator.

$$f(\{u_1, x_1^1,\ldots,x_1^m\},\ldots\{u_n, x_n^1,\ldots,x_n^m\}) = \left(\sum_{j=1}^{n} w_j b_j^3\right)^{1/3}. \quad (35)$$

- If $\lambda = -1$, we get the harmonic IOWAS (IOWHAS) operator.

$$f(\{u_1, x_1^1,\ldots,x_1^m\},\ldots\{u_n, x_n^1,\ldots,x_n^m\}) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{b_j}}. \quad (36)$$

- If $\lambda \to 0$, we get the geometric IOWAS (IOWGAS) operator.

$$f(\{u_1, x_1^1,\ldots,x_1^m\},\ldots\{u_n, x_n^1,\ldots,x_n^m\}) = \prod_{j=1}^{n} b_j^{w_j}. \quad (37)$$

Note that a lot of other particular cases could be studied following the OWA literature [8].

V. Numerical Example

In this Section, let us briefly analyze the calculation process of the IOWAS operator. Assume we have two sets of 5 arguments as follows: $X = (20, 50, 60, 30, 80)$ and $Y = (70, 50, 30, 90, 60)$. In this case, consider the following order-inducing variables: $U = (6, 8, 3, 9, 4)$. The weighting vector used in the aggregation is as follows: $W = (0.3, 0.2, 0.2, 0.2, 0.1)$.

With this information, we can aggregate the IOWAS operator in the following way.

First, we calculate the individual sums between the sets X and Y.

- $(x_1 + y_1) = (20 + 70) = 90$.
- $(x_2 + y_2) = (50 + 50) = 100$.
- $(x_3 + y_3) = (60 + 30) = 90$.
- $(x_4 + y_4) = (30 + 90) = 120$.
- $(x_5 + y_5) = (80 + 60) = 140$.

Next, we reorder the data according to the order-inducing variables in a decreasing way.

- $u_4 > u_2 > u_1 > u_5 > u_3$.

Therefore, the aggregation of the IOWAS operator proceeds as follows.

IOWAS = $0.3 \times 120 + 0.2 \times 100 + 0.2 \times 90 + 0.2 \times 140 + 0.1 \times 90 = 111$.

As we can see, the IOWAS operator follows a very similar methodology to the IOWA operator [5] but with an additional first step that calculates the individual sums between the arguments of the two sets X and Y.

Note that this methodology is very useful in a wide range of situations because it is very common in the real world to make aggregations of sums. For example, in business and economics it is very useful to calculate the average values of different economic variables such as the costs and the quantity of product. Usually the total costs depend on several costs that are summed, especially the fixed cost plus the variable cost. If we consider different sections, companies, cities or countries, we may calculate the average cost in each region. Therefore, we are aggregating a set of sums. And obviously, in these situations the IOWAS operator may be very useful to under or overestimate the information according to a complex attitudinal character.

VI. Conclusions

This paper has studied averaging sums with induced aggregation operators. By doing so, the aggregation process of sums can be assessed with complex attitudinal characters that depend on order-inducing variables. First, the study has introduced the IOWAS operator. The main advantage of this approach is its flexibility to adapt to different environments when dealing with averaging sums. Further extensions have been developed forming the IGOWAs and the Quasi-IOWAS operator, respectively. These generalizations include the geometric IOWAS and the quadratic IOWAS operator.

The work has analyzed the applicability of the new approach and it is very broad because all the studies that use averaging sums can be revised with this new approach. Note that averaging sums are very common when dealing with several sets of elements. For example, in economics and business it can represent the average sum of the sales of a set of products, the costs, and the demand. The work has briefly showed a simple numerical example in order to understand the new approach.

In future research, additional improvements are possible by using other issues in the aggregation including distances [30-31], Choquet integrals [32], interval numbers [33-34], fuzzy numbers [35-36] and moving averages [19]. Moreover, it is relevant to consider different types of ordering process of the arguments since the paper focuses on ordering after the sum although it is also possible to order before the individual sums. A deeper focus on the applicability will be developed by considering more research topics. Attention will be given to economic problems that deal with average sums.

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