

On fuzzy relations, adjunctions, and functional fuzzy relations

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Abstract—The problem of studying when a given a fuzzy relation can be characterized in terms of a fuzzy function is revisited. Although several authors have published works on the ‘functionality’ of a fuzzy relation, we focus on a particular version which, to the best of our knowledge, fits better for the definition of adjunction between generalized fuzzy structures.

I. INTRODUCTION

The notion of adjunction (also called isotone Galois connection) is an algebraic tool which enables to establish bridges between different research areas and has found a number of applications, both theoretical and practical. The interested reader can find more details for instance in [1], [2].

We continue our research line on the construction of adjunctions within different environments. To begin with, it is worth to note that the well-known adjunction theorem by Freyd, which characterises the existence of adjoint, does not apply to any of the cases that we consider in that the theorem is stated between *homogeneous* structures (both the domain and the codomain have the same algebraic structure, either a preordered set, a poset, a fuzzy order, etc), but we are interested in knowing whether, given a mapping f from a certain structured set A to an unstructured set B , it is possible *both* to provide B with the corresponding algebraic structure *and* to construct a mapping g which is the right adjoint to f with respect to the newly defined structure.

Several results have been already obtained in this respect: in [3], our underlying environment was that of crisp functions between a poset (resp. preordered set) and an unstructured set; then, in [4], the paradigm was shifted to the fuzzy case, considering the corresponding problem in which the set A has fuzzy preposet structure; and more recently, in [5], we considered in addition fuzzy equivalence relations generalizing the equality, both in A and in B .

In this work, we start to consider a further step of abstraction, since the fuzzy extensions given in [4] and [5] lack of fuzziness precisely on the adjunction, namely, both mappings f and g are crisp, and we would like to work with a really fuzzy version of the notion of adjunction, in which f and g are fuzzy functions.

The paper is structured as follows: in Section II, we introduce the preliminary notions which will be needed thereafter; then, in Section III we introduce the notion of completely functional fuzzy relation together with an alternative characterization; later, Section IV focuses on the morphisms and

partial functions in the sense of Gottwald and, once again, a characterization result is presented; finally, Section V introduces a new notion of fuzzy adjunction whose components are indeed fuzzy functions.

II. PRELIMINARIES

The most usual underlying structure for considering fuzzy extensions of Galois connections is that of complete residuated lattice, $\mathbb{L} = (L, \leq, \top, \perp, \otimes, \rightarrow)$. As usual, supremum and infimum will be denoted by $\vee$ and $\wedge$ respectively, and we will write $\alpha \leftrightarrow \beta$ as an abbreviation of $(\alpha \rightarrow \beta) \wedge (\beta \rightarrow \alpha)$ for $\alpha, \beta \in L$.

An $\mathbb{L}$ -fuzzy set is a mapping from the universe set to the membership values structure $X: U \rightarrow L$ where $X(u)$ means the degree in which u belongs to X . Given X and Y two $\mathbb{L}$ -fuzzy sets, X is said to *be included in* Y , denoted as $X \subseteq Y$, if $X(u) \leq Y(u)$ for all $u \in U$.

An $\mathbb{L}$ -fuzzy binary relation between A and B is an $\mathbb{L}$ -fuzzy subset of $A \times B$, that is, a mapping $\mu: A \times B \rightarrow L$. Given μ , its domain and its image are defined as follows:

- $\text{dom}(\mu) = \{a \in A \mid \text{there exists } b \in B \text{ such that } \mu(a, b) = \top\}$
- $\text{im}(\mu) = \{b \in B \mid \text{there exists } a \in A \text{ such that } \mu(a, b) = \top\}$

An $\mathbb{L}$ -fuzzy binary relation on U is an $\mathbb{L}$ -fuzzy subset of $U \times U$, that is $R_U: U \times U \rightarrow L$, and it is said to be:

- *Reflexive* if $R_U(a, a) = \top$ for all $a \in U$.
- $\otimes$ -*Transitive* if $R_U(a, b) \otimes R_U(b, c) \leq R_U(a, c)$ for all $a, b, c \in U$.
- *Symmetric* if $R_U(a, b) = R_U(b, a)$ for all $a, b \in U$.

From now on, when no confusion arises, we will omit the prefix “ $\mathbb{L}$ -”.

Definition 1: A *fuzzy preposet* is a pair $\mathbb{A} = \langle A, \rho_A \rangle$ in which ρ_A is a reflexive and $\otimes$ -transitive fuzzy relation on A .

Definition 2: A fuzzy relation $\approx$ on A is said to be a:

- *Fuzzy equivalence relation* if $\approx$ is a reflexive, $\otimes$ -transitive and symmetric fuzzy relation on A .
- *Fuzzy equality* if $\approx$ is a fuzzy equivalence relation satisfying that $\approx(a, b) = \top$ implies $a = b$, for all $a, b \in A$.

Notation 1: We will use the infix notation for a fuzzy equivalence relation, that is: for $\approx: A \times A \rightarrow L$ a fuzzy equivalence relation, we denote $a_1 \approx a_2$ to refer to $\approx(a_1, a_2)$.

Definition 3: For a fuzzy equivalence relation $\approx: A \times A \rightarrow L$, the equivalence class of an element $a \in A$ is a fuzzy set $[a]_{\approx}: A \rightarrow L$ defined by $[a]_{\approx}(u) = (a \approx u)$ for all $u \in A$.

Remark 1: Note that $[x]_{\approx} = [y]_{\approx}$ if and only if $(x \approx y) = \top$: on the one hand, if $[x]_{\approx} = [y]_{\approx}$, then $(x \approx y) = [x]_{\approx}(y) = [y]_{\approx}(y) = \top$, by reflexive property; conversely, if $(x \approx y) = \top$, then $[x]_{\approx}(u) = (x \approx u) = (y \approx x) \otimes (x \approx u) \leq (y \approx u) = [y]_{\approx}(u)$, for all $u \in A$.

Definition 4: Given a universe U , a fuzzy partition of U is a family Π of fuzzy subsets of U such that

- i) for all $u \in U$ there exists $A \in \Pi$ such that $A(u) = \top$
- ii) for all $A \in \Pi$ there exists $u \in U$ such that $A(u) = \top$
- iii) $A(u) \otimes B(u) \leq \bigwedge_{x \in U} A(x) \leftrightarrow B(x)$, for all $A, B \in \Pi$ and $u \in U$

Notation 2: Similarly to definition 3, given $\mu: A \times B \rightarrow L$, for all $a \in A$ we will consider the fuzzy sets $\mu_a: B \rightarrow L$ defined by $\mu_a(b) = \mu(a, b)$.

Definition 5: A fuzzy structure $\mathcal{A} = \langle A, \approx_A \rangle$ is a set A endowed with a fuzzy equivalence relation $\approx_A$.

From now on, for a fuzzy structure $\mathcal{A}$, the underlying set and the fuzzy equivalence relation are denoted by A and $\approx_A$ respectively.

A number of different approaches to the notion of fuzzy function can be found in the literature. The main problem with the definition resides in that the fuzziness of the function would imply that the function itself should be a fuzzy set (in some sense). This difficulty can be overcome by means of the use of suitable fuzzy equivalences in the domain and the codomain of the function.

The first approach we will consider is that in which the equality is substituted by fuzzy equivalence relations in both the domain and the codomain, i.e., considering a compatible mapping between fuzzy structures. This is a standard definition [6] for which we will use the term “morphism” (between fuzzy structures) in order to simplify the statements of the related results.

Definition 6: A morphism between two fuzzy structures $\mathcal{A}$ and $\mathcal{B}$ is a mapping $f: A \rightarrow B$ which is compatible with $\approx_A$ and $\approx_B$ i.e. for all $a_1, a_2 \in A$ the following inequality holds $(a_1 \approx_A a_2) \leq (f(a_1) \approx_B f(a_2))$. In this case, we write $f: \mathcal{A} \rightarrow \mathcal{B}$.

An alternative approach is given directly in terms of fuzzy relations [7], [8]:

Definition 7: Let $\mathcal{A} = \langle A, \approx_A \rangle$ and $\mathcal{B} = \langle B, \approx_B \rangle$ be fuzzy structures. A partial fuzzy function from $\mathcal{A}$ to $\mathcal{B}$ is a fuzzy relation $\mu: A \times B \rightarrow L$ satisfying the following conditions:

- (Ext1) $\mu(a_1, b) \otimes (a_1 \approx_A a_2) \leq \mu(a_2, b)$ for all $a_1, a_2 \in A$ and $b \in B$.
- (Ext2) $\mu(a, b_1) \otimes (b_1 \approx_B b_2) \leq \mu(a, b_2)$ for all $a \in A$ and $b_1, b_2 \in B$.
- (Part) $\mu(a, b_1) \otimes \mu(a, b_2) \leq (b_1 \approx_B b_2)$ for all $a \in A$ and $b_1, b_2 \in B$

Moreover, μ is said to be a perfect fuzzy function whenever the following condition holds:

(Tot) For all $a \in A$ there exists $b \in B$ satisfying that $\mu(a, b) = \top$.

The two definitions above are closely related as follows:

Given a morphism $f: \langle A, \approx_A \rangle \rightarrow \langle B, \approx_B \rangle$ between fuzzy structures, there exists a perfect fuzzy function $\mu: A \times B \rightarrow L$ defined by $\mu(a, b) = (f(a) \approx_B b)$ for all $a \in A$ and $b \in B$ such that $\mu(a, f(a)) = \top$. On the other hand, given a perfect fuzzy function $\mu: A \times B \rightarrow L$, every mapping $f: A \rightarrow B$ satisfying $\mu(a, f(a)) = \top$ is a morphism between $\langle A, \approx_A \rangle$ and $\langle B, \approx_B \rangle$.

The following theorem states that, when $\approx_B$ is a fuzzy equality, there exists a one-to-one correspondence between f 's and μ 's.

Theorem 1 (See [6, pg. 188]): Let $\approx_A$ be a fuzzy equivalence on A and let $\approx_B$ be a fuzzy equality on B . There exists a bijection between perfect fuzzy functions and morphisms.

III. FUZZY RELATIONS VS MORPHISMS BETWEEN FUZZY STRUCTURES

In order to characterize which fuzzy relations do correspond to any of the different definitions of a fuzzy function, given a morphism $f: \langle A, \approx_A \rangle \rightarrow \langle B, \approx_B \rangle$, it is possible to define a fuzzy relation $\mu: A \times B \rightarrow L$ by $\mu(a, b) = (f(a) \approx_B b)$. The problem that we consider in this section is the opposite: given a fuzzy relation μ , we study when it is possible to define a morphism f between fuzzy structures such that $\mu(a, b) = (f(a) \approx_B b)$.

Such a problem does not always have a solution even in the crisp case (in which the fuzzy equivalences $\approx$ are replaced by equivalence relations $\equiv$) as the following examples show:

Example 1: Consider the relations $\mu_1, \mu_2: \{a_1, a_2\} \times \{b_1, b_2\} \rightarrow \mathbb{L}$, where $\mathbb{L} = (L, \perp, \top, \cdot, \rightarrow)$ is a non-trivial residuated lattice, defined below:

μ_1	b_1	b_2	μ_2	b_1	b_2
a_1	$\top$	$\top$	a_1	$\top$	$\top$
a_2	$\perp$	$\perp$	a_2	$\perp$	$\top$

In neither of the previous examples it is possible to define a mapping $f: \{a_1, a_2\} \rightarrow \{b_1, b_2\}$ and an equivalence relation $\equiv_B$ on B such that $\mu_i(a, b) = (f(a) \equiv_B b)$: in a such case, in one hand we have

$$\perp = \mu_1(a_2, f(a_2)) = (f(a_2) \equiv f(a_2)) = \top$$

which yields a contradiction; on the other hand, by transitive property

$(b_1 \equiv f(a_1)) \otimes (f(a_1) \equiv b_2) \otimes (b_2 \equiv f(a_2)) \leq (b_1 \equiv f(a_2))$ but $\mu(a_1, b_1) \otimes \mu(a_1, b_2) \otimes \mu(a_2, b_2) = \top$ and $\mu(a_2, b_1) = \perp$ which also implies a contradiction.

It is not difficult to observe that the previous examples failed because the families $\{\mu_i(a_1), \mu_i(a_2)\}$ do not form a partition in B and, hence, no equivalence relation $\equiv_B$ can be defined on B such that $\mu_i(a, b) = (f(a) \equiv_B b)$ holds.

When considering the previous approach in the fuzzy setting, it is natural to rephrase it in terms of fuzzy partitions. Actually, we will see below that if the set of fuzzy sets given by $\{\mu_a\}_{a \in A}$ is a fuzzy partition, then there is a solution for

our problem and, hence, it is a sufficient condition. Before proceeding with the formal result, we introduce the notion of ‘functionality’ of a fuzzy relation as follows:

Definition 8: Let $\mu \in L^{A \times B}$ be a fuzzy relation. We say that μ is *completely functional* if there exist two fuzzy structures $\mathcal{A} = \langle A, \approx_A \rangle$ and $\mathcal{B} = \langle B, \approx_B \rangle$ and a morphism $f: \mathcal{A} \rightarrow \mathcal{B}$ such that $\mu(a, b) = (f(a) \approx_B b)$ for all $a \in A$ and $b \in B$.

Now, we can state the sufficient condition stated above:

Proposition 1: Let $\mu \in L^{A \times B}$ be a fuzzy relation. If $\{\mu_a\}_{a \in A}$ is a fuzzy partition on B , then μ is completely functional.

The following example shows that the condition in the above proposition is not necessary.

Example 2: Let $A = \{a_1, a_2\}$ and $B = \{b_1, b_2, b_3, b_4\}$ be two sets and let $\mu: A \times B \rightarrow [0, 1]$ be the following fuzzy relation between A and B :

μ	b_1	b_2	b_3	b_4
a_1	1	0.1	1	0.8
a_2	0.1	1	0.1	0.1

Let $f: A \rightarrow B$ be the mapping defined as $f(a_1) = b_1, f(a_2) = b_2$ and the fuzzy equivalence in B given by

$\approx_B$	b_1	b_2	b_3	b_4
b_1	1	0.1	1	0.8
b_2	0.1	1	0.1	0.1
b_3	1	0.1	1	0.8
b_4	0.8	0.1	0.8	1

Observe that $\mu(a_i, b_j) = (f(a_i) \approx b_j)$ for all $i \in \{1, 2\}$ and $j \in \{1, 2, 3, 4\}$. However, the sets of fuzzy sets $\{\mu_{a_1}, \mu_{a_2}\}$ is not a fuzzy partition on B since for $b_4 \in B$ neither $\mu(a_1, b_4) = 1$ nor $\mu(a_2, b_4) = 1$.

It is remarkable that the condition in Proposition 1 does not impose any restriction with respect to the fuzzy structure $\mathcal{A}$. The following result clarifies this issue.

Lemma 1: A fuzzy relation $\mu \in L^{A \times B}$ is completely functional if and only if there exist a fuzzy structure $\mathcal{B} = \langle B, \approx_B \rangle$ and a morphism $f: \langle A, = \rangle \rightarrow \mathcal{B}$ such that $\mu(a, b) = (f(a) \approx_B b)$ for all $a \in A$ and $b \in B$.

The main result in this section characterizes the fuzzy relations which are completely functional.

Theorem 2: Let $\mu \in L^{A \times B}$ be a fuzzy relation. Then μ is completely functional if and only if the following conditions hold:

- 1) For each $a_1, a_2 \in A$ and $b_1, b_2 \in B$ the following inequality holds

$$\mu(a_1, b_1) \otimes \mu(a_2, b_1) \otimes \mu(a_1, b_2) \leq \mu(a_2, b_2) \quad (1)$$

- 2) $\text{dom}(\mu) = A$.

Proof: Assume that $\mu \in L^{A \times B}$ is completely functional and let $f: \langle A, \approx_A \rangle \rightarrow \langle B, \approx_B \rangle$ be a morphism such that $\mu(a, b) = (f(a) \approx_B b)$, for all $a \in A, b \in B$. Since $\approx_B$

is transitive, for $a_1, a_2 \in A$ and $b_1, b_2 \in B$:

$$\begin{aligned} & \mu(a_1, b_1) \otimes \mu(a_2, b_1) \otimes \mu(a_1, b_2) \\ &= (f(a_1) \approx_B b_1) \otimes (f(a_2) \approx_B b_1) \otimes (f(a_1) \approx_B b_2) \\ &= (f(a_2) \approx_B b_1) \otimes (b_1 \approx_B f(a_1)) \otimes (f(a_1) \approx_B b_2) \\ &\leq (f(a_2) \approx_B b_2) = \mu(a_2, b_2) \end{aligned}$$

Moreover, for any $a \in A$, since $\approx_B$ is reflexive, $\mu(a, f(a)) = (f(a) \approx_B f(a)) = \top$ which means that μ_a is normal and $a \in \text{dom}(\mu)$.

Conversely, let $\mu \in L^{A \times B}$ be a fuzzy relation for which Conditions 1. and 2. hold. Define a fuzzy binary relation $\approx_\mu$ on B as follows:

$$(b_1 \approx_\mu b_2) = (b_1 = b_2) \vee \bigvee_{a \in A} (\mu(a, b_1) \otimes \mu(a, b_2))$$

The relation satisfies the properties of reflexivity and symmetry. For the $\otimes$ -transitivity, given $b_1, b_2, b_3 \in B$, by distributivity of $\vee$ with respect to $\otimes$, we have

$$\begin{aligned} & (b_1 \approx_\mu b_2) \otimes (b_2 \approx_\mu b_3) \\ &= \left[(b_1 = b_2) \otimes (b_2 = b_3) \right] \vee \\ & \vee \left[(b_1 = b_2) \otimes \bigvee_{a \in A} (\mu(a, b_2) \otimes \mu(a, b_3)) \right] \\ & \vee \left[(b_2 = b_3) \otimes \bigvee_{a \in A} (\mu(a, b_1) \otimes \mu(a, b_2)) \right] \\ & \vee \left[\bigvee_{a \in A} (\mu(a, b_1) \otimes \mu(a, b_2)) \otimes \bigvee_{a \in A} (\mu(a, b_2) \otimes \mu(a, b_3)) \right] \end{aligned}$$

The first three components of the supremum above are clearly less than $(b_1 \approx_\mu b_3)$, therefore we have just to prove the inequality for the fourth component. By applying distributivity once again we obtain

$$\begin{aligned} & \bigvee_{a \in A} (\mu(a, b_1) \otimes \mu(a, b_2)) \otimes \bigvee_{a \in A} (\mu(a, b_2) \otimes \mu(a, b_3)) = \\ &= \bigvee_{a \in A} \bigvee_{a' \in A} (\mu(a, b_1) \otimes \mu(a, b_2) \otimes \mu(a', b_2) \otimes \mu(a', b_3)) \\ &\stackrel{(1)}{\leq} \bigvee_{a' \in A} (\mu(a', b_1) \otimes \mu(a', b_3)) \leq (b_1 \approx_\mu b_3) \end{aligned}$$

As a result, $\approx_\mu$ is a fuzzy equivalence relation.

For $a \in A$, since μ_a is normal and by the axiom of choice, it is possible to define $f: A \rightarrow B$ such that $\mu(a, f(a)) = \top$, for each $a \in A$. Firstly, observe that f is a morphism between $\langle A, = \rangle$ and $\langle B, \approx_\mu \rangle$ because $\approx_\mu$ is reflexive.

Now, we have just to prove that $\mu(a, b) = (f(a) \approx_\mu b)$ for all $a \in A, b \in B$. The equality trivially holds in the case $f(a) = b$.

If $f(a) \neq b$ we will prove the two inequalities. Firstly,

$$\begin{aligned} (f(a) \approx_\mu b) &= \bigvee_{a' \in A} (\mu(a', f(a)) \otimes \mu(a', b)) \\ &= \bigvee_{a' \in A} (\mu(a', f(a)) \otimes \mu(a', b) \otimes \mu(a, f(a))) \\ &\stackrel{(1)}{\leq} \bigvee_{a' \in A} \mu(a, b) = \mu(a, b) \end{aligned}$$

Conversely

$$\begin{aligned}\mu(a, b) &= \mu(a, f(a)) \otimes (a, b) \\ &\leq \bigvee_{a' \in A} (\mu(a', f(a)) \otimes \mu(a', b)) = (f(a) \approx_\mu b)\end{aligned}$$

As a consequence of the previous result, it is not difficult to prove that a fuzzy relation μ is completely functional if and only if the family of fuzzy sets $\{\mu_a\}_{a \in A}$ satisfies the second and third conditions of fuzzy partition (see Definition 4). This is reasonable in that the first condition of fuzzy partition would imply that our underlying mapping f should be surjective, which is more than what we are requiring.

Furthermore, the role of condition 2 in the previous theorem is exclusively related to the fact that f is a total function (defined on every element $a \in A$), and condition (1) is the actual key of the ‘functionality’ of μ . This justifies the introduction of the following definition

Definition 9: A fuzzy relation $\mu: A \times B \rightarrow L$ is said to be *functional* if condition (1) is satisfied.

It is worth to remark that the transitivity for $\approx_\mu$ in the proof of Theorem 2 only depends on condition (1), hence it is a fuzzy equivalence relation provided μ is functional (not necessarily completely functional).

Definition 10: Given a functional fuzzy relation μ , the fuzzy equivalence relation $\approx_\mu$ defined for $b_1, b_2 \in B$ by

$$(b_1 \approx_\mu b_2) = (b_1 = b_2) \vee \bigvee_{a \in A} (\mu(a, b_1) \otimes \mu(a, b_2))$$

will be called the *fuzzy equivalence induced by μ on B* .

Note that, if $b_1 \in \text{im}(\mu)$, then $(b_1 \approx_\mu b_2) = \mu(a_1, b_2)$ for any $a_1 \in A$ such that $f(a_1) = b_1$ (or, equivalently, $\mu(a_1, b_1) = \top$). For the proof, we have to consider just the case in which $b_1 \neq b_2$, hence

$$\begin{aligned}(b_1 \approx_\mu b_2) &= \bigvee_{a \in A} (\mu(a, b_1) \otimes \mu(a, b_2)) \\ &= \bigvee_{a \in A} (\mu(a, b_1) \otimes \mu(a, b_2) \otimes \mu(a_1, b_1)) \\ &\stackrel{(1)}{\leq} \bigvee_{a \in A} \mu(a_1, b_2) = \mu(a_1, b_2)\end{aligned}$$

For the other inequality

$$\begin{aligned}\mu(a_1, b_2) &= \mu(a_1, b_1) \otimes \mu(a_1, b_2) \\ &\leq \bigvee_{a \in A} (\mu(a, b_1) \otimes \mu(a, b_2)) = (b_1 \approx_\mu b_2)\end{aligned}$$

By symmetry of the construction, if $b_2 \in \text{im}(\mu)$, then $(b_1 \approx_\mu b_2) = \mu(a_2, b_1)$ for any $a_2 \in A$ such that $f(a_2) = b_2$. In fact, $\approx_\mu$ is the least fuzzy equivalence relation satisfying the two previous equalities, therefore we have the following result:

Proposition 2: Given a completely functional fuzzy relation μ , the fuzzy equivalence relation induced by μ on B , $\approx_\mu$, is

the transitive closure of the fuzzy relation

$$(b_1 \approx_B b_2) = \begin{cases} \mu(a_1, b_2) & \text{if there exists } a_1 \in A \\ & \text{such that } \mu(a_1, b_1) = \top; \\ \mu(a_2, b_1) & \text{if there exists } a_2 \in A \\ & \text{such that } \mu(a_2, b_2) = \top; \\ \top & \text{if } b_1 = b_2; \\ \perp & \text{otherwise.} \end{cases}$$

Furthermore, the fuzzy equivalence relation $\approx_\mu$ is the least one on B which enables to define μ in terms of a morphism between fuzzy structures. Formally,

Proposition 3: Given a completely functional fuzzy relation $\mu: A \times B \rightarrow L$, then for any possible morphism between fuzzy structures $f: \langle A, \approx_A \rangle \rightarrow \langle B, \approx_B \rangle$ satisfying $\mu(a, b) = (f(a) \approx_B b)$, we have that $\approx_B$ includes $\approx_\mu$.

IV. FUZZY RELATIONS VS PERFECT AND PARTIAL FUZZY FUNCTIONS

We are interested now in studying the link between a fuzzy relation and another suitable notion of fuzzy function, namely, the perfect and the partial fuzzy functions introduced by Gottwald.

To begin with, we obtain that the notion of complete functionality of a fuzzy relation given in the previous section coincides, in some sense, with the existence of a perfect function between suitable fuzzy structures. The formal statement is the following:

Theorem 3: A fuzzy relation $\mu \in L^{A \times B}$ is completely functional if and only if there exist two fuzzy equivalence relations $\approx_A$ on A and $\approx_B$ on B such that μ is a perfect fuzzy function between $\langle A, \approx_A \rangle$ and $\langle B, \approx_B \rangle$.

Proof: Let us assume that μ is completely functional, then there exists a fuzzy equivalence relation $\approx_B$ on B and a mapping $f: A \rightarrow B$ such that $\mu(a, b) = (f(a) \approx_B b)$; and let us prove that μ is a perfect fuzzy function between $\langle A, \approx_A \rangle$ and $\langle B, \approx_B \rangle$.

We have to check that all the conditions in Definition 7 hold:

(Ext1) Trivial, since $\mu(a_1, b) \otimes (a_1 = a_2) \leq \mu(a_2, b)$.

(Ext2) By hypothesis and $\otimes$ -transitivity of $\approx_B$

$$\begin{aligned}\mu(a, b_1) \otimes (b_1 \approx_B b_2) &= (f(a) \approx_B b_1) \otimes (b_1 \approx_B b_2) \\ &\leq (f(a) \approx_B b_2) = \mu(a, b_2).\end{aligned}$$

(Part) By hypothesis, and symmetry and $\otimes$ -transitivity of $\approx_B$

$$\begin{aligned}\mu(a, b_1) \otimes \mu(a, b_2) &= (f(a) \approx_B b_1) \otimes (f(a) \approx_B b_2) \\ &= (b_1 \approx_B f(a)) \otimes (f(a) \approx_B b_2) \\ &\leq (b_1 \approx_B b_2).\end{aligned}$$

(Tot) Directly by Theorem 2 (item 1).

Conversely, let us assume that μ is a perfect fuzzy function between two fuzzy structures $\langle A, \approx_A \rangle$ and $\langle B, \approx_B \rangle$, and let us prove that μ is functional, in fact, we will show the conditions in Theorem 2.

To begin with, condition **(Tot)** implies that μ_a is normal for all $a \in A$; therefore the second condition holds.

For condition (1), given $a_1, a_2 \in A$ and $b_1, b_2 \in B$, by **(Part)** and **(Ext2)** we have that

$$\begin{aligned} \mu(a_1, b_1) \otimes \mu(a_2, b_1) \otimes \mu(a_1, b_2) &\leq \mu(a_2, b_1) \otimes (b_1 \approx_B b_2) \\ &\leq \mu(a_2, b_2) \end{aligned}$$

As stated above, condition (1) somehow encodes the ‘functionality’ of a fuzzy relation μ , since if the first condition is missing ($\text{dom}(\mu) \neq A$ but still $\text{dom}(\mu) \neq \emptyset$) then we still retain ‘partial functionality’. The formal development of this idea is given in the rest of this section.

Theorem 4: Given a fuzzy relation $\mu: A \times B \rightarrow L$, the following conditions are equivalent:

- 1) μ is functional.
- 2) μ is a partial fuzzy function from $\langle A, \approx_A \rangle$ to $\langle B, \approx_B \rangle$ for certain $\approx_A$ and $\approx_B$.
- 3) μ^{-1} is a partial fuzzy function from $\langle B, \approx_B \rangle$ to $\langle A, \approx_A \rangle$ for certain $\approx_A$ and $\approx_B$.
- 4) μ and μ^{-1} are partial fuzzy functions between $\langle A, \approx_A \rangle$ and $\langle B, \approx_B \rangle$ for certain $\approx_A$ and $\approx_B$.

The previous result can be rephrased as well in terms of morphisms provided that we restrict our attention to the domain of μ . In fact, we have a two-directional interpretation in which it is possible to define two morphisms $f: \langle \text{dom}(\mu), \approx_{\mu^{-1}} \rangle \rightarrow \langle B, \approx_B \rangle$ and $g: \langle \text{im}(\mu), \approx_\mu \rangle \rightarrow \langle A, \approx_A \rangle$ satisfying

$$(f(a) \approx_\mu b) = \mu(a, b) = (a \approx_{\mu^{-1}} g(b)) \quad (2)$$

for all $a \in \text{dom}(\mu)$ and $b \in \text{im}(\mu)$.

It is remarkable that equations (2) can be read as the properties required for the pair (f, g) to be a Galois connection between $\text{dom}(\mu)$ and $\text{im}(\mu)$. Indeed, the fact that the underlying relations are fuzzy equivalences and, in particular, fuzzy preorders, implies that (f, g) is actually an adjunction, co-adjunction, and a (left) and right-Galois connection [2] and, hence, one has

$$((f \circ g)(b) \approx_\mu b) = \top \quad \text{and} \quad ((g \circ f)(a) \approx_{\mu^{-1}} a) = \top$$

Contrariwise to the behaviour of the crisp descriptions in the previous section, the following example shows that, when neither $\text{dom}(\mu) = A$ nor $\text{im}(\mu) = B$, the crisp descriptions associated to a fuzzy relation μ do not allow, in general, to reconstruct μ .

Example 3: Let L be an arbitrary complete residuated lattice. Consider as well the fuzzy relation $\mu: \{a, b\} \times \{c, d\} \rightarrow L$ defined as follows for some fixed elements $\alpha, \beta, \gamma \in L$:

$$\begin{array}{c|cc} \mu & c & d \\ \hline a & \top & \beta \\ b & \alpha & \gamma \end{array}$$

It is a matter of straight computation to check that the fuzzy relation μ is functional if and only if the following inequalities hold:

- 1) $\mu(a, c) \otimes \mu(b, c) \otimes \mu(a, d) = \top \otimes \alpha \otimes \beta \leq \gamma = \mu(b, d)$.
- 2) $\mu(a, d) \otimes \mu(b, d) \otimes \mu(a, c) = \beta \otimes \gamma \otimes \top \leq \alpha = \mu(b, c)$.
- 3) $\mu(b, c) \otimes \mu(a, c) \otimes \mu(b, d) = \alpha \otimes \top \otimes \gamma \leq \beta = \mu(a, d)$.

As a result μ is functional if and only if $\alpha \otimes \beta \leq \gamma \leq \alpha \leftrightarrow \beta$,

which is a not very restrictive condition in that there are lots of particular examples satisfying them.

The fuzzy equivalences induced by μ are the following ones:

$$\begin{array}{c|cc} \approx_\mu & c & d \\ \hline c & \top & \beta \\ d & \beta & \top \end{array} \quad \begin{array}{c|cc} \approx_{\mu^{-1}} & a & b \\ \hline a & \top & \alpha \\ b & \alpha & \top \end{array}$$

Since $\text{dom}(\mu) = \{a\}$ and $\text{im}(\mu) = \{c\}$, we have that the crisp descriptions associated to μ must be defined by $f(a) = c$ and $g(c) = a$.

Now, by equations (2), it is not possible to reconstruct the value of $\mu(b, d)$ since $b \notin \text{dom}(\mu)$ and $d \notin \text{im}(\mu)$.

V. FUZZY RELATIONS AND FUZZY ADJUNCTIONS

In this section, we introduce a novel definition of fuzzy adjunction in which the role of left and right adjoints is played by arbitrary fuzzy relations.

Definition 11: Let $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ be fuzzy preposets and $\mu: A \times B \rightarrow L$ and $\nu: B \times A \rightarrow L$ be fuzzy relations. The pair (μ, ν) is said to be a fuzzy adjunction between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ if the following conditions hold:

- (Ad1) For all $a_1 \in A$ and $b_1 \in B$ there exist $a_2 \in A$ and $b_2 \in B$ such that $\mu(a_1, b_1) \leq \mu(a_1, b_2) \otimes \nu(b_2, a_2)$ and $\nu(b_1, a_1) \leq \nu(b_1, a_2) \otimes \mu(a_2, b_2)$.
- (Ad2) For all $a_1, a_2 \in A$ and $b_1, b_2 \in B$, one has
 - i) $\rho_A(a_1, a_2) \otimes \mu(a_1, b_1) \otimes \nu(b_2, a_2) \leq \rho_B(b_1, b_2)$.
 - ii) $\rho_B(b_1, b_2) \otimes \mu(a_1, b_1) \otimes \nu(b_2, a_2) \leq \rho_A(a_1, a_2)$.

This definition is coherent in some sense with our previous approaches about adjunctions in a fuzzy setting.

Definition 12 ([9]): Let $\approx_A$ be a fuzzy equivalence relation on A . A fuzzy binary relation $\rho_A: A \times A \rightarrow L$ is said to be

- (i) $\approx_A$ -*reflexive* if $(a_1 \approx_A a_2) \leq \rho_A(a_1, a_2)$ for all $a_1, a_2 \in A$.
- (ii) $\otimes$ - $\approx_A$ -*antisymmetric* if $\rho_A(a_1, a_2) \otimes \rho_A(a_2, a_1) \leq (a_1 \approx_A a_2)$ for all $a_1, a_2 \in A$.

Definition 13: Given a fuzzy structure $\mathcal{A} = \langle A, \approx_A \rangle$, the pair $\mathbb{A} = \langle \mathcal{A}, \rho_A \rangle$ will be called a $\otimes$ - $\approx_A$ -*fuzzy preordered structure* or simply *fuzzy preordered structure* (when there is no risk of confusion), if ρ_A is a fuzzy relation that is $\approx_A$ -reflexive, $\otimes$ - $\approx_A$ -antisymmetric and $\otimes$ -transitive.

Definition 14: [5] Let $\mathbb{A}$ and $\mathbb{B}$ be two fuzzy preordered structures. Given two morphisms $f: \mathcal{A} \rightarrow \mathcal{B}$ and $g: \mathcal{B} \rightarrow \mathcal{A}$, the pair (f, g) is said to be an *adjunction* between $\mathbb{A}$ and $\mathbb{B}$ (briefly, $(f, g): \mathbb{A} \rightleftarrows \mathbb{B}$) if $\rho_A(a, g(b)) = \rho_B(f(a), b)$ for all $a \in A$ and $b \in B$.

Proposition 4: Let $\mathbb{A}$ and $\mathbb{B}$ be two fuzzy preordered structures and let $\mu: A \times B \rightarrow L$ and $\nu: B \times A \rightarrow L$ be completely functional fuzzy relations for which there exist morphisms $f: \mathcal{A} \rightarrow \mathcal{B}$ and $g: \mathcal{B} \rightarrow \mathcal{A}$ such that $\mu(a, b) = (f(a) \approx_B b)$ and $\nu(b, a) = (g(b) \approx_A a)$ for all $a \in A$ and $b \in B$.

The pair (μ, ν) is a fuzzy adjunction between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ if and only if (f, g) is an adjunction between the fuzzy preordered structures $\langle A, \approx_A, \rho_A \rangle$ and $\langle B, \approx_B, \rho_B \rangle$.

From the previous definition, the following proposition can be obtained straightforwardly.

Proposition 5: Let $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ be fuzzy preposets and $\mu: A \times B \rightarrow \mathbb{L}$ and $\nu: B \times A \rightarrow \mathbb{L}$ be fuzzy relations.

- 1) If μ and ν are total (i.e. $\text{dom}(\mu) = A$ and $\text{dom}(\nu) = B$), the pair (μ, ν) satisfies the condition (Ad1).
- 2) If the pair (μ, ν) satisfies the condition (Ad1) then, for all $a_1 \in A$ and $b_1 \in B$, there exists $a_2 \in A$ such that $\mu(a_1, b_1) = \mu(a_1, b_1) \otimes \nu(b_1, a_2) \otimes \nu(b_1, a_2)$.
- 3) If the pair (μ, ν) satisfies the condition (Ad1) then, for all $a_1 \in A$ and $b_1 \in B$, there exists $b_2 \in B$ such that $\nu(b_1, a_1) = \nu(b_1, a_1) \otimes \mu(a_1, b_2) \otimes \mu(a_1, b_2)$.

In order to study the properties of fuzzy adjunctions, we need to introduce some preliminary definitions, inspired on the similar ones already given for morphisms.

Definition 15: Let $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ be fuzzy preposets. A fuzzy relation $\mu: A \times B \rightarrow \mathbb{L}$ is said to be *isotone* if $\rho_A(a_1, a_2) \otimes \mu(a_1, b_1) \otimes \mu(a_2, b_2) \leq \rho_B(b_1, b_2)$ for all $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Proposition 6: Let $\mathbb{A}$ and $\mathbb{B}$ be two fuzzy preordered structures and let $\mu: A \times B \rightarrow \mathbb{L}$ be a completely functional fuzzy relation for which there exists a morphism $f: \mathcal{A} \rightarrow \mathcal{B}$ such that $\mu(a, b) = (f(a) \approx_B b)$ for all $a \in A$ and $b \in B$. The relation μ is isotone if and only if the morphism f is isotone (i.e. $\rho_A(a_1, a_2) \leq \rho_B(f(a_1), f(a_2))$).

Definition 16: Let $\langle A, \rho_A \rangle$ be a fuzzy preposet. A fuzzy relation $\mu: A \times A \rightarrow \mathbb{L}$ is said to be:

- *inflationary* if $\mu(a_1, a_2) \leq \rho_A(a_1, a_2)$ for all $a_1, a_2 \in A$.
- *deflationary* if $\mu(a_1, a_2) \leq \rho_A(a_2, a_1)$ for all $a_1, a_2 \in A$.

Proposition 7: Let $\mathbb{A}$ be a fuzzy preordered structure and let $\mu: A \times A \rightarrow \mathbb{L}$ be a completely functional fuzzy relation for which there exists a morphism $f: \mathcal{A} \rightarrow \mathcal{A}$ such that $\mu(a_1, a_2) = (f(a_1) \approx_B a_2)$ for all $a_1, a_2 \in A$. The relation μ is inflationary (deflationary, resp.) if and only if the morphism f is inflationary (i.e. $\rho_A(a_1, f(a_1)) = \top$) (deflationary, resp.).

The following theorem gives a first characterization of fuzzy adjunctions, which resembles the classical behavior of crisp adjunctions.

Theorem 5: Let $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$ be fuzzy preposets and $\mu: A \times B \rightarrow \mathbb{L}$ and $\nu: B \times A \rightarrow \mathbb{L}$ be fuzzy relations such that the pair (μ, ν) satisfies the condition (Ad1). Then, (μ, ν) is a fuzzy adjunction if and only if μ and ν are isotone, $\nu \circ \mu$ is inflationary and $\mu \circ \nu$ is deflationary.

Proof: Consider that (μ, ν) is a fuzzy adjunction between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$.

- The following sequence proves that $\nu \circ \mu$ is inflationary by using reflexivity of ρ_B and the condition (Ad2):

$$\begin{aligned} (\nu \circ \mu)(a_1, a_2) &= \bigvee_{b \in B} \mu(a_1, b) \otimes \nu(b, a_2) \\ &= \bigvee_{b \in B} \rho_B(b, b) \otimes \mu(a_1, b) \otimes \nu(b, a_2) \\ &\leq \rho_A(a_1, a_2) \end{aligned}$$

- Now, we similarly prove that $\mu \circ \nu$ is deflationary:

$$\begin{aligned} (\mu \circ \nu)(b_1, b_2) &= \bigvee_{a \in A} \nu(b_1, a) \otimes \mu(a, b_2) \\ &= \bigvee_{a \in A} \rho_A(a, a) \otimes \nu(b_1, a) \otimes \mu(a, b_2) \\ &\leq \rho_B(b_2, b_1) \end{aligned}$$

- In order to prove the isotonicity of μ , first we prove that, for all $a_1, a_2 \in A$ and $b \in B$, there exists $a_3 \in A$ with

$$\rho_A(a_1, a_2) \otimes \mu(a_2, b) \leq \rho_A(a_1, a_3) \otimes \nu(b, a_3) \quad (3)$$

By Proposition 5, the fact that $\nu \circ \mu$ is inflationary and transitivity of ρ_A , one has:

$$\begin{aligned} \rho_A(a_1, a_2) \otimes \mu(a_2, b) &= \rho_A(a_1, a_2) \otimes \mu(a_2, b) \otimes \nu(b, a_3) \otimes \nu(b, a_3) \\ &\leq \rho_A(a_1, a_2) \otimes \rho_A(a_2, a_3) \otimes \nu(b, a_3) \\ &\leq \rho_A(a_1, a_3) \otimes \nu(b, a_3) \end{aligned}$$

Now, by using (3) and the condition (Ad2), one has:

$$\begin{aligned} \rho_A(a_1, a_2) \otimes \mu(a_1, b_1) \otimes \mu(a_2, b_2) &\leq \rho_A(a_1, a_3) \otimes \mu(a_1, b_1) \otimes \nu(b_2, a_3) \\ &\leq \rho_B(b_1, b_2) \end{aligned}$$

- The proof for the isotonicity of ν is analogous.

Contrariwise, assume that (μ, ν) satisfies the condition (Ad1), μ and ν are isotone, $\nu \circ \mu$ is inflationary and $\mu \circ \nu$ is deflationary. First, we prove item b) in condition (Ad2) by using Proposition 5, the facts that $\nu \circ \mu$ is inflationary and ν is isotone, and transitivity of ρ_A :

$$\begin{aligned} \rho_B(b_1, b_2) \otimes \mu(a_1, b_1) \otimes \nu(b_2, a_2) &= \rho_B(b_1, b_2) \otimes \mu(a_1, b_1) \otimes \nu(b_1, a_3) \otimes \nu(b_1, a_3) \otimes \nu(b_2, a_2) \\ &\leq \rho_B(b_1, b_2) \otimes \rho_A(a_1, a_3) \otimes \nu(b_1, a_3) \otimes \nu(b_2, a_2) \\ &\leq \rho_A(a_1, a_3) \otimes \rho_A(a_3, a_2) \leq \rho_A(a_1, a_2) \end{aligned}$$

Item a) in condition (Ad2) is analogously proved. ■

Corollary 1: Let (μ, ν) be a fuzzy adjunction between $\langle A, \rho_A \rangle$ and $\langle B, \rho_B \rangle$. The following conditions hold:

- 1) $\mu(a, b_1) \otimes (\mu \circ \nu \circ \mu)(a, b_2) \leq \rho_B(b_1, b_2) \wedge \rho_B(b_2, b_1)$ for all $a \in A$ and $b_1, b_2 \in B$.
- 2) $\nu(b, a_1) \otimes (\nu \circ \mu \circ \nu)(b, a_2) \leq \rho_A(a_1, a_2) \wedge \rho_A(a_2, a_1)$ for all $a_1, a_2 \in A$ and $b \in B$.
- 3) For all $a_1 \in A$ and $b \in B$ there exists $a_2 \in A$ such that $\mu(a_1, b) \leq \rho_A(a_1, a_2)$.
- 4) For all $a \in A$ and $b_1 \in B$ there exists $b_2 \in B$ such that $\nu(b_1, a) \leq \rho_B(b_2, b_1)$.

VI. RELATED APPROACHES

There are several papers dealing with different notions of fuzzy function, and their interrelations: one of the first approaches was given by Sasaki [10], and one of the most well-known is that of Demirci [7], [11]; we can even find recent works in which the different notions existing in the literature are further compared and put into context, see [12] and [13]. A similar characterization of functional fuzzy relations in terms of partial fuzzy functions was already stated in [14], using the

notion of uniform fuzzy relation and uniform F -function, and a completely different motivation. We prefer to stick to our approach since, at least in the context of our research topic, it is much more natural to consider a functional fuzzy relation than a uniform fuzzy relation.

Concerning the generalization to the notion of adjunction to the fuzzy case, to the best of our knowledge, the first approach was due to Bělohlávek [15]. Later, a number of publications have introduced different approaches to either fuzzy adjunctions or fuzzy Galois connections, see [16], [17], [18], [19], [20]. The latter notion of fuzzy Galois connection introduced by Yao in [20] was used in our previous work [4], where we were interested in constructing a right adjoint (or residual mapping) associated to a given mapping $f: \langle A, \rho_A \rangle \rightarrow B$ from a fuzzy preposet $\langle A, \rho_A \rangle$ into an unstructured set B . The fact the mappings in this approach are crisp rather than fuzzy motivated the search for the use of fuzzy functions and lead to the notion introduced in this work.

VII. CONCLUSIONS AND FURTHER WORK

We have revisited the problem of studying when a given a fuzzy relation can be characterized in terms of a fuzzy function. Then, we have provided a notion of fuzzy adjunction as a pair of (completely functional) fuzzy relations fulfilling certain properties which generalizes naturally the notion used in previous approaches (see [4] and [5]).

As future work, we are planning the characterization of existence of this type of fuzzy adjunctions in different fuzzy environments (preordered or partially ordered sets with or without corresponding fuzzy equivalence relations).

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