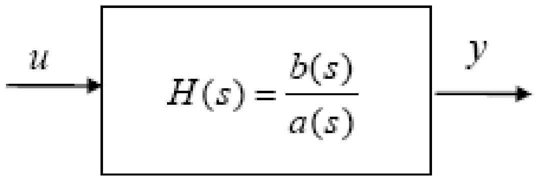
3.9 NATURAL RESPONSE vs. POLE LOCATIONS

Pole location

LECTURE 10

TF



$$H(s) = \frac{b(s)}{a(s)}$$

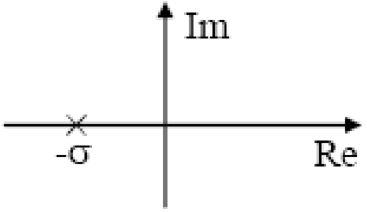
Response of unforced system

$(u = 0)$

Also called '**natural response**'

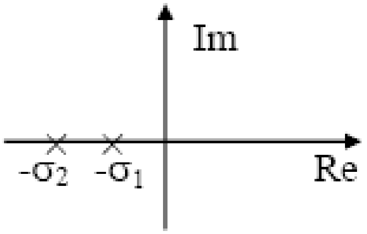
Can show that this has the same form as response to an impulse, i.e., impulse response.

[Recall, Poles: roots of a(s) = 0; Zeros: roots of b(s) = 0]



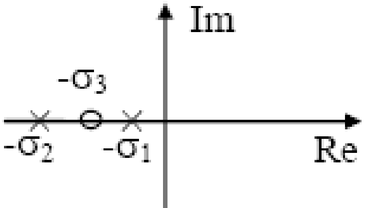
$$H(s) = \frac{1}{s+\sigma}$$

$$y(t) = c_1 e^{-\sigma t}$$



$$\frac{1}{(s+\sigma_1)(s+\sigma_2)}$$

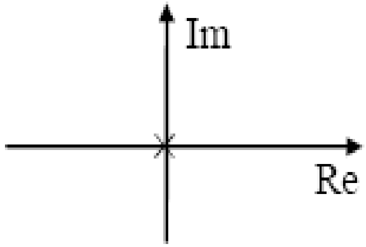
$$y(t) = c_1 e^{-\sigma_1 t} + c_2 e^{-\sigma_2 t}$$



$$\frac{s+\sigma_3}{(s+\sigma_1)(s+\sigma_2)}$$

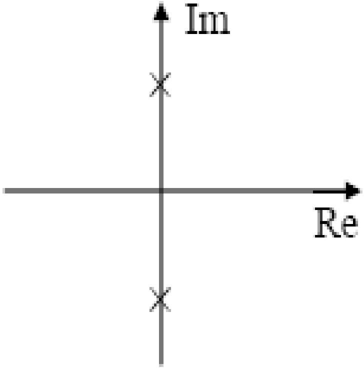
$$y(t) = c_3 e^{-\sigma_1 t} + c_4 e^{-\sigma_2 t}$$

(ADAPTED FROM SATISH NAIR)



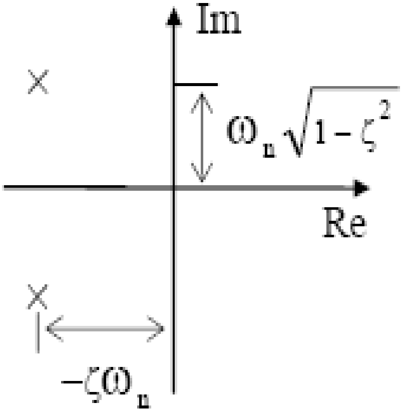
$$\frac{1}{s}$$

$$y(t) = y_o$$



$$\frac{1}{s^2 + \sigma}$$

$$y(t) = A\sin(\sqrt{\sigma}\,t) + B\cos(\sqrt{\sigma}\,t)$$



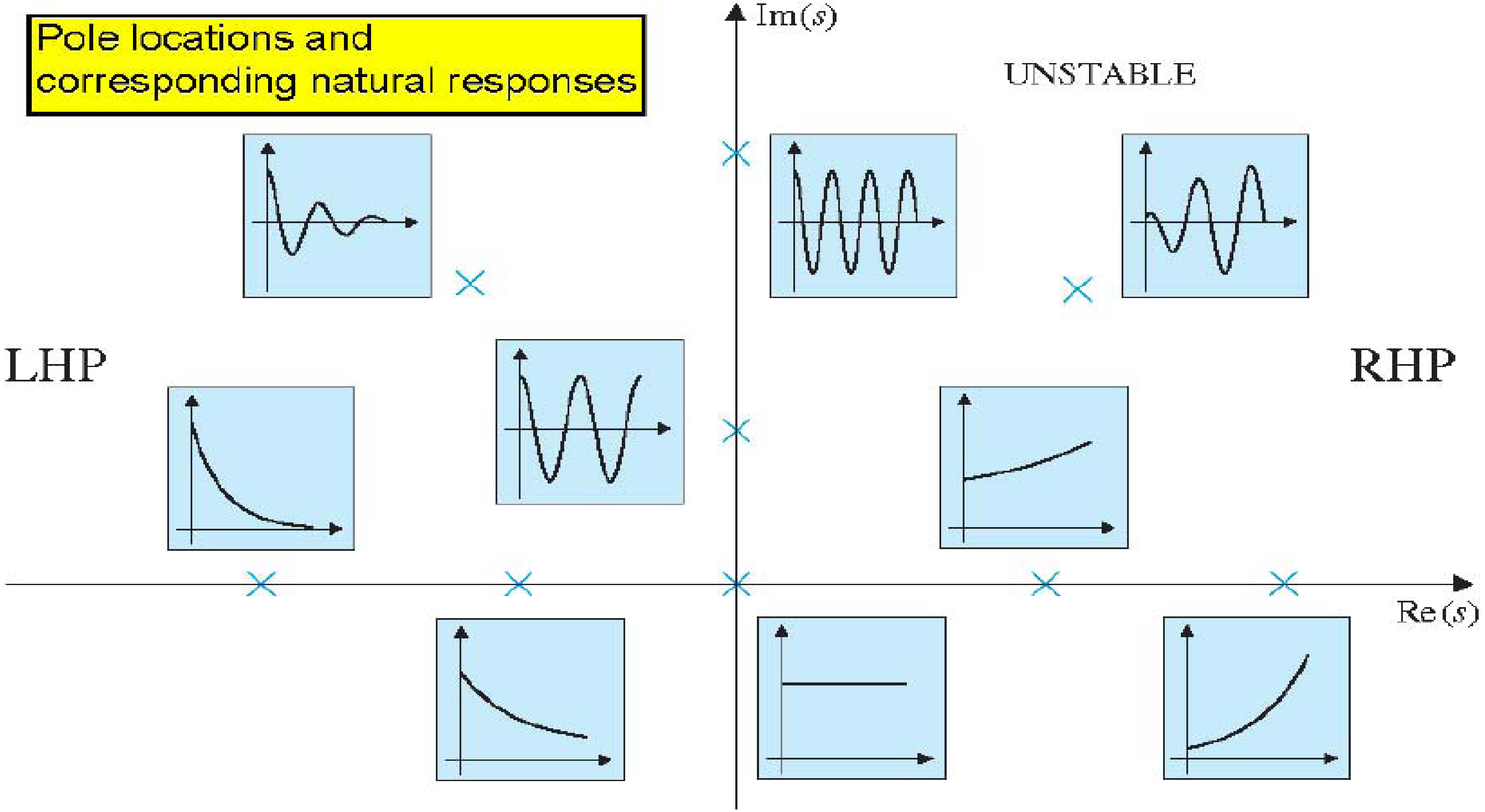
$$\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$y(t) = Ae^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t + \phi)$$

(Similar eqns. for RHP poles)

What do the corresponding time response plots look like?

Natural response (unforced response) has the same form as the impulse response!



Pole locations and corresponding natural responses

Analytical solution of a first order differential equation

Notation: $\dot{y}(t) \cong dy / dt$

Consider two different first order dynamics, $\dot{y}(t) = a$ and $\dot{y}(t) = a * y$

Identify real world 'systems' that exhibit such time dynamics for positive and negative values of a.

How does a first order system evolve in time? (i.e., what is its response?)

Any differential equation of the type $a_1\dot{y}(t) + a_2 y(t) = a_3 u(t)$, with initial condition $y(0) = y_0$, can be recast in the 'standard' form below by dividing each term by a_2, to obtain

$$\tau\dot{y}(t) + y(t) = Ku(t)\,; \quad y(0) = y_0$$

where the time constant $\tau = a_1 / a_2$, and the gain $K = a_3 / a_2$, and $u(t)$ is the time dependent input.

If the input is a unit step input, i.e., $u(t) = 1.0$, then it can be used to shown that $\boxed{y(t) = y_0 e^{-t/\tau} + K(1 - e^{-t/\tau})}$.

What happens when $t = \tau$? Substituting $t = \tau$, we get $y(\tau) =$

$y_0 e^{-1} + K(1 - e^{-1}) = y_0 - y_0 + y_0 e^{-1} + K(1 - e^{-1}) = y_0 + (K - y_0)(1 - e^{-1}) = y_0 + (K - y_0)0.632$.

This shows how the system reaches 63% of the 'change' in value from initial value (i.e., K-y_0) in one time constant. Remember that the system started at a non-zero value of y_0 in this case.

So, for a first order system, time constant is the time it takes for the output to reach 63.2% of the 'change'.

Example: What is the response of the system $2\dot{x}(t) + 3x(t) = 5u(t)$ with $u(t) = 2.0$, and $x(0)=10$?

Solution: The first step is to convert the system into its standard form. This yields $(2/3)\dot{x}(t) + x(t) = (5/3)u(t)$. The time constant is 2/3, and the gain is 5/3. From the form of the solution given above, we can find the response in this particular case as $x(t) = x_0 e^{-t/\tau} + K(1 - e^{-t/\tau})$ which in our case becomes

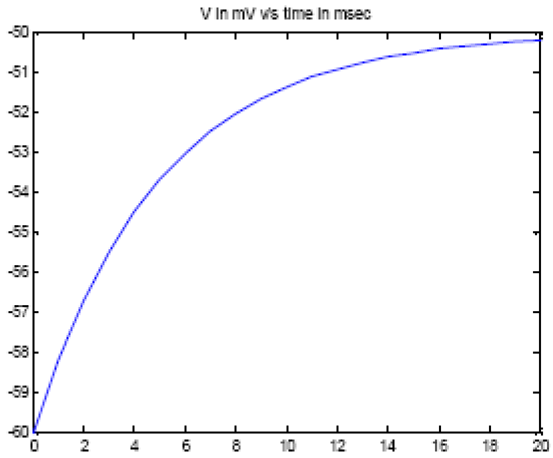
$x(t) = 10 * e^{-3t/2} + (10/3) * (1 - e^{-t/\tau})$. Does this satisfy the initial condition? $x(0)=10 - 0 = 10$……..so, YES.

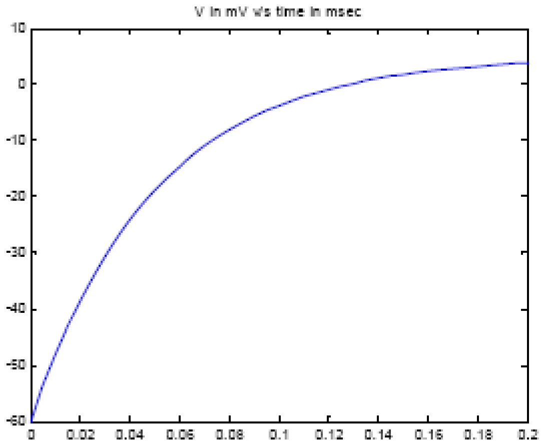
Exercise: What is the response of the system $3\dot{v}(t) + 5v(t) = u(t)$ with $u(t)=5.0$, and $v(0)=0$? Also, solve and plot using MATLAB and verify your answer.

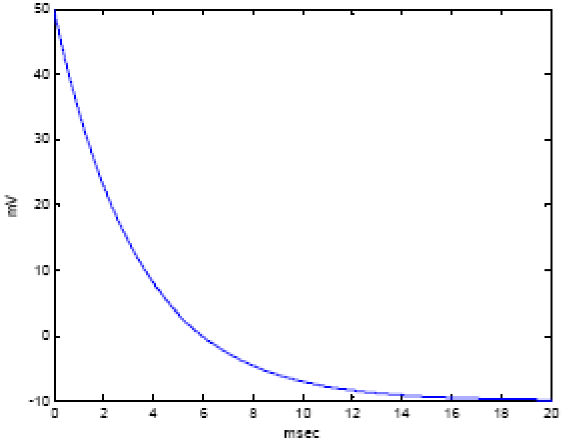
Meaning of time constant? Time constant for a first order system is the time it takes for the output to reach 63.2% of the 'change'. Estimate the time constants for the systems shown below.

RECALL: $y(\tau)=y_0+(K-y_0)0.632$

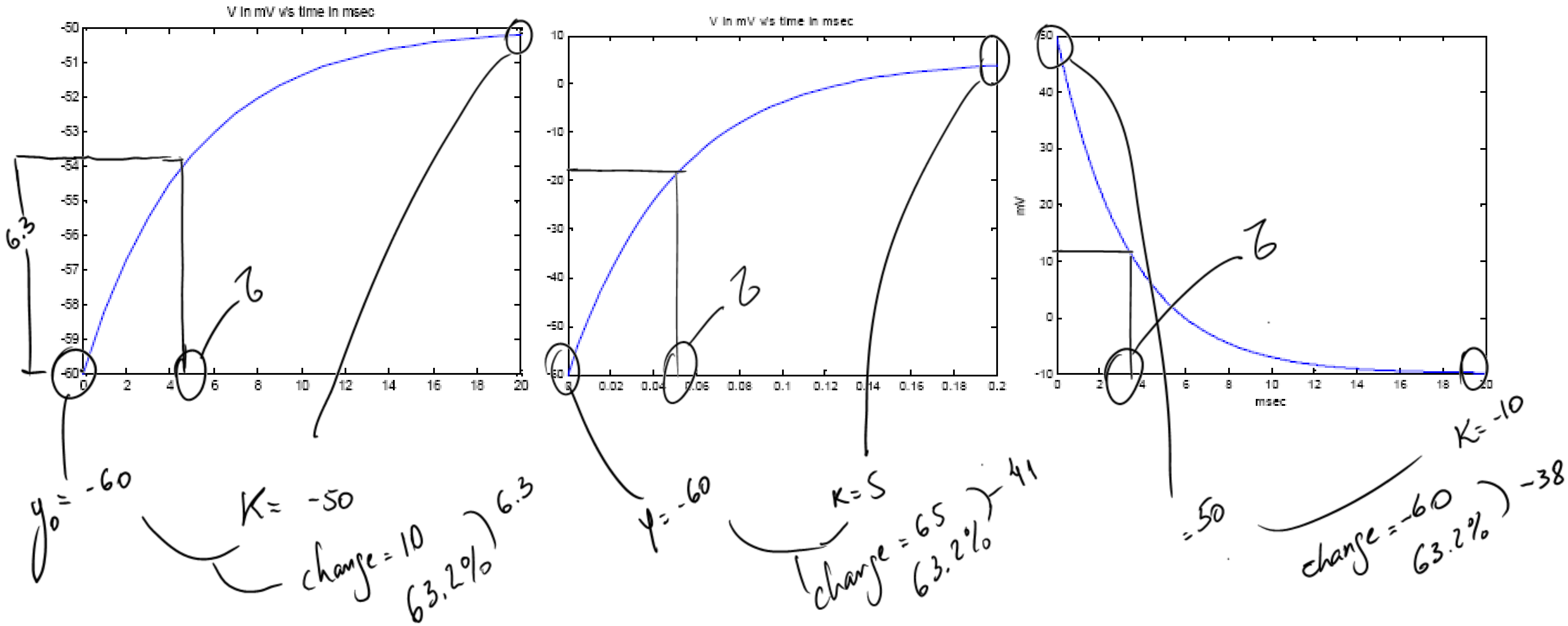
$y(t)=y_0 e^{-t/\tau}+K(1-e^{-t/\tau})$







<u>Meaning of time constant</u>? Time constant for a first order system is the time it takes for the output to reach 63.2% of the ‘change’. Estimate the time constants for the systems shown below.



6.3

τ

$y_0 = -60$

$K = -50$

change = 10

63.2% → 6.3

τ

$y = -60$

$K = 5$

change = 65

63.2% → 41

τ

= 50

$K = -10$

change = -60

63.2% → -38

Analytical Solutions for General First and Second Order Systems

First Order System:

$\tau\dot{y} + y = a \qquad y(0) = y_0$

Constant a means a step input.

Homogeneous solution:

$\tau\dot{y} + y = 0$

Assume $y_h = c_1 e^{\lambda t} \quad \Rightarrow \dot{y}_h = \lambda c_1 e^{\lambda t}$

$\tau\lambda c_1 e^{\lambda t} + c_1 e^{\lambda t} = 0 \quad \Rightarrow \lambda = -1/\tau$

$\therefore \quad y_h = c_1 e^{-t/\tau}$

Particular solution:

Assume $y_p = c_2 \quad \Rightarrow \dot{y}_p = 0$

$\tau\dot{y}_p + y_p = a \quad \Rightarrow c_2 = a$

Total solution: $y = y_h + y_p$

$y = c_1 e^{-t/\tau} + a$

$y|_{t=0} = y_0 \quad \Rightarrow y_0 = c_1 + a$

$\therefore \quad c_1 = y_0 - a$

$\therefore \quad y = (y_0 - a)e^{-t/\tau} + a \quad = y_0 e^{-t/\tau} + a(1 - e^{-t/\tau})$

MATLAB commands

```
syms tau a
y=dsolve('tau*Dy+y=a,y(0)=y0','t');
pretty(y)
```

Second Order System:

$M\ddot{y} + D\dot{y} + Ky = F$

The procedure of solving this equation is omitted here. Define

$\omega_n = \sqrt{\frac{K}{M}}$, and $\zeta = \frac{D}{2M\omega_n}$

where ω_n is the *undamped natural frequency*, and ζ is the *damping ratio*.

Depending on whether ζ is less than, greater than or equal to unity, three solution cases are possible

Case 1: Underdamped system ($0 < \zeta < 1$)

$$y(t) = e^{-\zeta\omega_n t}\left[\frac{\zeta\omega_n y(0) + \dot{y}(0)}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}) + x(0)\cos(\omega_n\sqrt{1-\zeta^2}\,t)\right] + \frac{F}{K}$$

Case 2: Critically damped system ($\zeta = 1$)

$$y(t) = e^{-\omega_n t}\left[x(0) + (\omega_n x(0) + \dot{x}(0)t\right] + \frac{F}{K}$$

Case 3: Over damped system ($\zeta > 1$)

$$y(t) = \frac{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n x(0) - \dot{x}(0)}{2\omega_n\sqrt{\zeta^2 - 1}} e^{-\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + \frac{\left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n x(0) + \dot{x}(0)}{2\omega_n\sqrt{\zeta^2 - 1}} e^{-\left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} + \frac{F}{K}$$

MATLAB commands

```
syms wn zeta y y0 y_dot0
y=dsolve('D2y+2*wn*zeta*Dy+wn^2*y=0,y(0)=y0,Dy(0)=y_dot0','t');
pretty(y)
```

Solve the first order ODE

$$\dot{V} + 3V = -30$$
$$V(0) = -60\ mV$$

that models a membrane with constant current injection starting at time t=0. Plot for 2 ms.

Analytical solution using MATLAB

```
>>V=dsolve('DV+3*V=-30,V(0)=-60','t')
V =
-10-50*exp(-3*t)
```

Note: This is the ‘exact’ solution using Matlab.

Numerical Method

(how any computer program would do it)

$$\dot{V}(t) \approx \frac{V(t+\Delta t) - V(t)}{\Delta t} \quad \text{(approximation used)}$$

$$V(t+\Delta t) \approx V(t) + \dot{V}(t) * \Delta t$$

Comparison of Solution Techniques

Example: $\dot{z}+3z=1$, $z(0)=1$

Laplace Transform Method

$$sZ(s)-1+3Z(s)=\frac{1}{s}$$

$$s^2Z(s)-s+3sZ(s)=1$$

$$Z(s)=\frac{s+1}{s^2+3s}$$

$$=\frac{s+1}{s(s+3)}$$

$$=\frac{1/3}{s}+\frac{2/3}{s+3}$$

$$\therefore z(t)=\frac{1}{3}+\frac{2}{3}e^{-3t}$$

MATLAB Commands

```
syms s
Z=(s+1)/(s^2+3*s);
z=ilaplace(Z)
```

Homogeneous (Complementary) /Particular Solution Method

$$\frac{dz}{dt}+3z=1$$

Homo. solution:

$$\frac{dz}{dt}+3z=0$$

Assume:

$$z=ce^{\lambda t}\Rightarrow\frac{dz}{dt}=\lambda ce^{\lambda t}$$

$$\lambda ce^{\lambda t}+3ce^{\lambda t}=0$$

$$\lambda=-3$$

$$\therefore z=ce^{-3t}$$

Non-homo. solution:

Assume

$$z=a\Rightarrow\frac{dz}{dt}=0$$

$$\frac{dz}{dt}+3z=1\Rightarrow 3a=1$$

$$\therefore a=\tfrac{1}{3}$$

Complete solution:

$$z=ce^{-3t}+\frac{1}{3}$$

$$z\,|_{t=0}=1\Rightarrow c=\tfrac{2}{3}$$

$$\therefore z(t)=\frac{2}{3}e^{-3t}+\frac{1}{3}$$

MATLAB Commands

```
z=dsolve(…
'Dz+3*z=1,z(0)=1','t')
;
pretty(z)
```

Numerical Method

(This how MATLAB does it.)

$$\dot{z}(t)\approx\frac{z(t+\Delta t)-z(t)}{\Delta t}$$

$$z(t+\Delta t)\approx z(t)+\dot{z}(t)*\Delta t$$

Given: $z(0)$ and t_{final}

Select: Δt

Pseudocode:

```
Initialize z and t
do
    $\dot{z}(t)=1-3z(t)$
    $z(t+\Delta t)=z(t)+\dot{z}(t)\Delta t$
    $t=t+\Delta t$
while $t\le t_{final}$
```

Sample calculation:

$z(0)=1$ and $t_{final}=2$

select $\Delta t=0.01$

step1:

$\dot{z}=1-3(1)=-2$

$z(.01)=1+(-2)*0.01=0.98$

$t=0+0.01$

step2:

$\dot{z}=1-3(.98)=-1.94$

$z(.02)=0.98+(-1.94)*0.01=0.9606$

$t=0.01+0.01$

step3:

$\dot{z}=1-3(0.9606)=-1.8818$

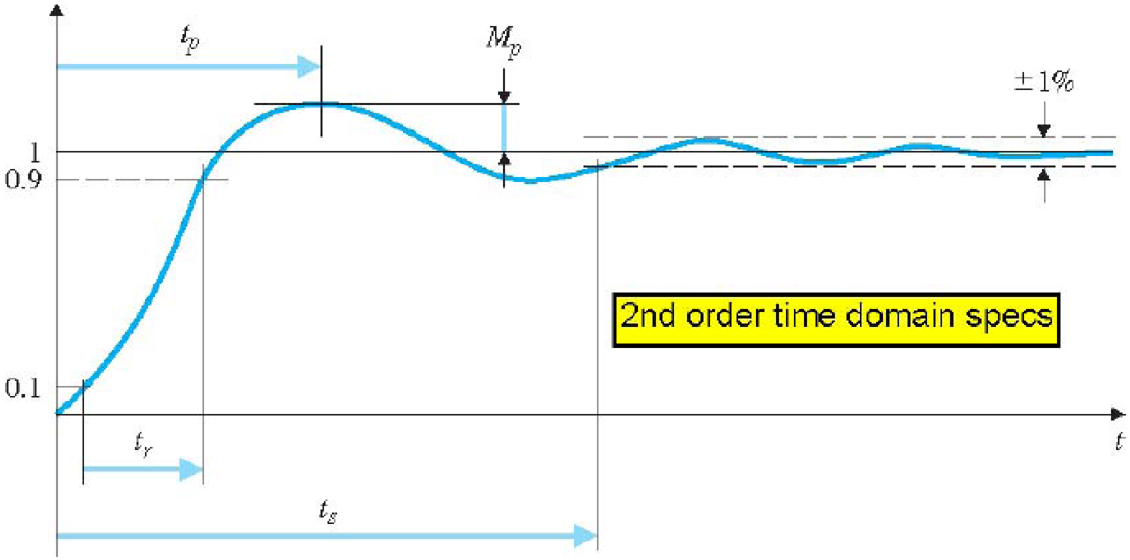
$z(.03)=0.9606+(-1.67)*0.01=0.941782$

$t=0.02+0.01$

$\vdots$

Note: This is the simple Euler's Method. Typically, more sophisticated methods, such as Euler-Cauchy method, Gil's method, and Runge-Kutta methods would yield more accurate results.

3.10 SECOND ORDER SYSTEMS CHARACTERISTICS FOR *DESIGN*



Consider H(s) = Y(s)/U(s) = $\omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, a ***general*** second order system with unit gain.

With zero ICs, let u(t) = 1(t) (unit step), what is the response y(t)?. Before we proceed, what are the pole locations for this system (coordinates?) and how does that help determine the form of the response?

Depending on whether ζ is less than, greater than or equal to unity, three solution cases are possible for y(t):

Case 1: Underdamped system ($0 < \zeta < 1$)

$y(t) = 1 - e^{-\sigma t}[\cos\omega_d t + (\sigma/\omega_d)\sin\omega_d t]$, where $\sigma = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Case 2: Critically damped system ($\zeta = 1$)

$y(t) = 1 - e^{-\omega_n t}(\omega_n t + 1)$

Case 3: Over damped system ($\zeta > 1$)

$y(t) = 1 + c_1 e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + c_2 e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$ *where* $c_1 = -(\omega_d + \sigma)/2\omega_d$ & $c_2 = -(\sigma - \omega_d)/2\omega_d$ For dominant second order systems (either 'pure' second order systems; or higher order systems where the extra poles and extra zeros are '4 times away' from the dominant poles, or if the closer 'extra' poles have zeros next to them leading to pole-zero cancellations)

*Formulae for some of transient specifications for the *underdamped* case are as follows:

Time Domain:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}; \qquad M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100\% \quad (\%OS = M_p * 100); \qquad \sigma = \zeta\omega_n$$

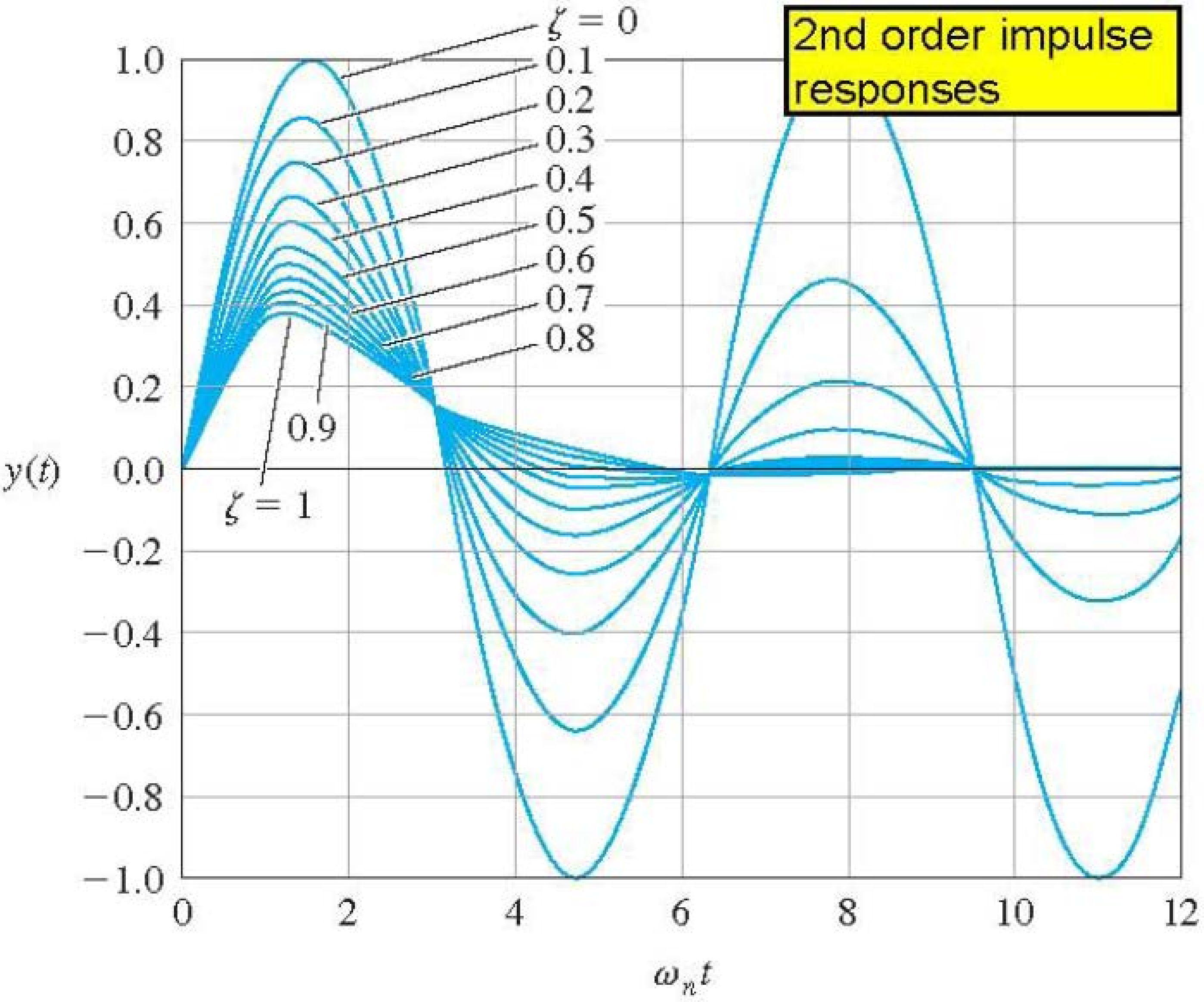
$$T_s = \frac{4.6}{\zeta\omega_n} \text{ (for 1\% settling band);} \quad T_r = \frac{1.8}{\omega_n} \text{ (for } \zeta = 0.5\text{);} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

$\omega_n \equiv$ undamped natural frequency; $\zeta \equiv$ damping ratio; $T_p \equiv$ peak time; $T_s \equiv$ settling time; $T_r \equiv$ rise time.

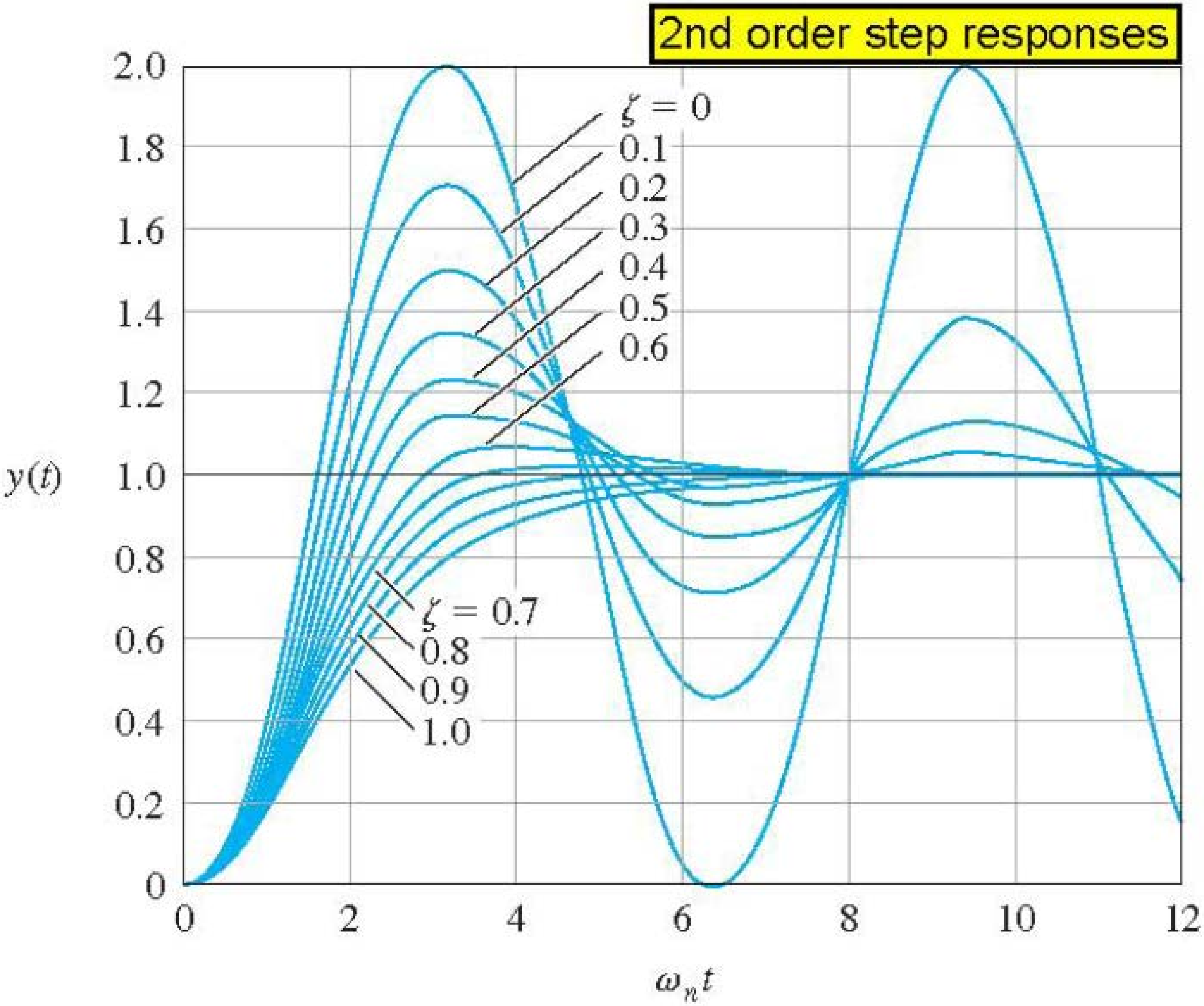
*How are these formulae modified when the system is not a dominant 2nd order system?
i.e., what is the effect of zeros and additional poles? Or, how do we infer something about systems of higher order with numerator terms also (not 'pure' second order)?

Approximations in the 'real world' for obtaining design equations: if 'pure' first or second order, these formulae hold, else the concept of 'dominant' roots holds……+ effect of additional poles and zeroes……..

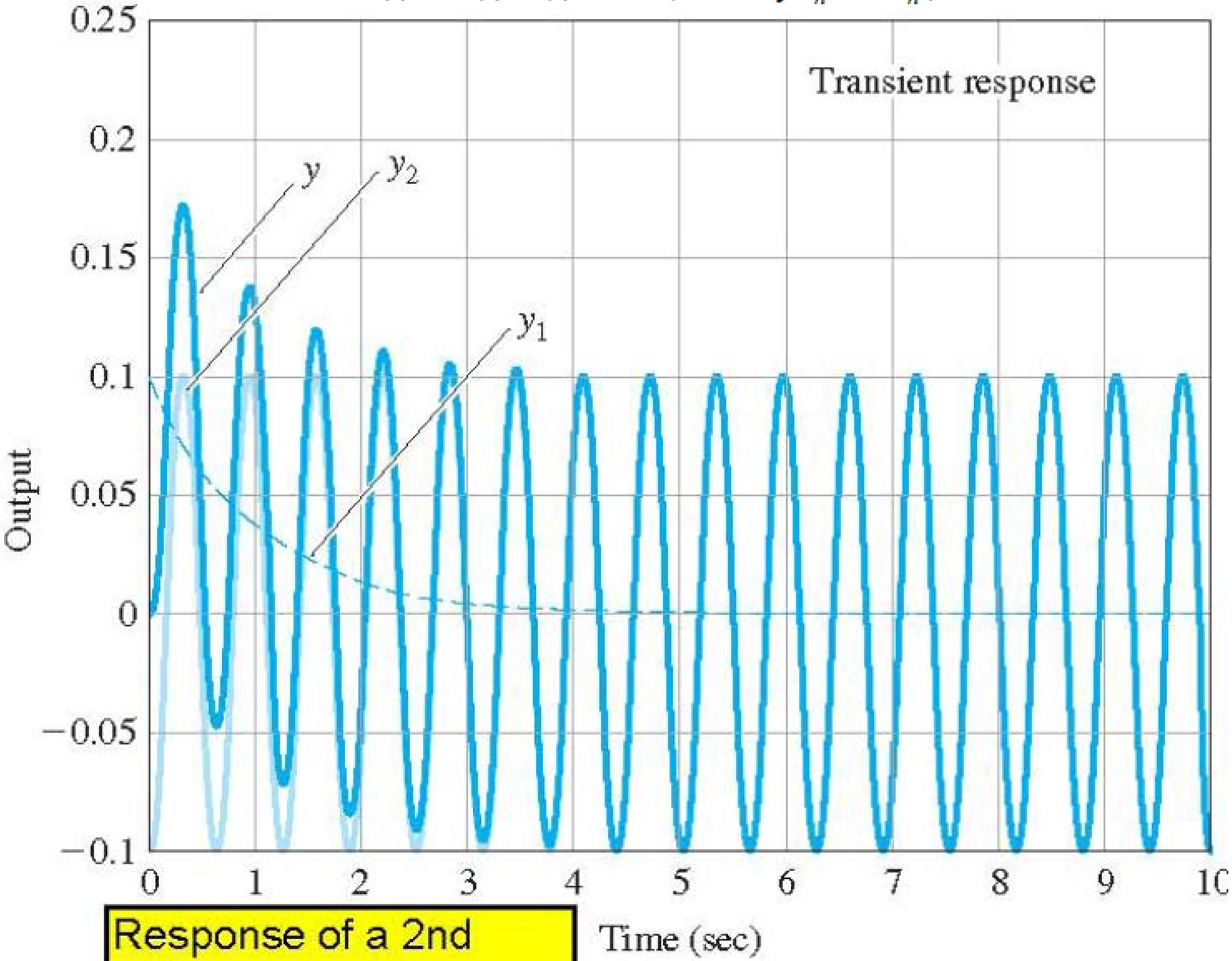
$$H(s) = Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$$



$$H(s) = Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$$



$$H(s) = Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$$



(a)

Response of a 2nd order system to a sinusoidal input

$$H(s) = Y(s)/U(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$$