LECTURE 11

## 3.11 CONVERTING TIME DOMAIN SPECS TO FREQ DOMAIN SPECS

Recall the time domain specifications for a 'pure' second order system: $t_r = \frac{1.8}{w_n}$ , $M_p = e^{-\frac{\varsigma\pi}{\sqrt{1-\varsigma^2}}}$ , $t_s = \frac{4.6}{\varsigma w_n}$

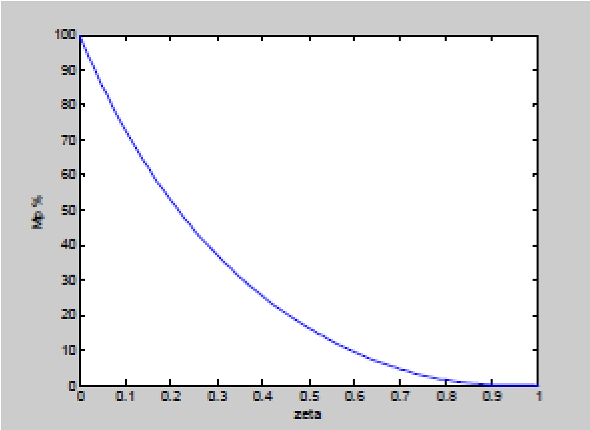
The requirements of the customer are $M_p < 5\%$, $t_r < 1\sec$ and $t_s < 4\sec$. As a controls engineer you need to convert these to frequency domain specifications. How would you do this?

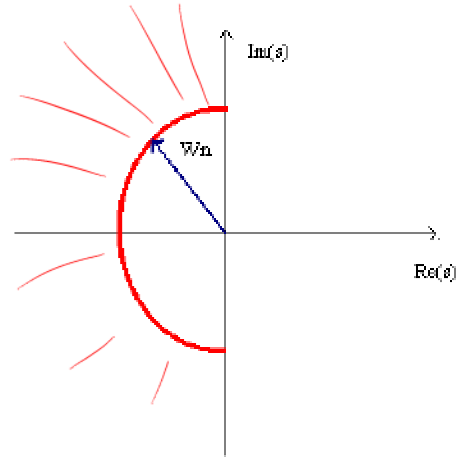
Using the formulae given above we can convert that into finding the frequency domain specifications as follows:

$w_n > \frac{1.8}{t_r}$ which makes the value of $w_n$ to be $w_n$ >1.8 as $t_r$ <1sec. Similarly for finding the value of $\zeta$ we use $\varsigma > \sqrt{\left(\log_e^{\frac{1}{M_p}} / \pi\right)^2 / (1 + \left(\log_e^{\frac{1}{M_p}} / \pi\right)^2)}$. As $M_p < 5\%$, i.e., 0.05, substitute the value in the above equation to get the value of $\varsigma > 0.6944$. Then finding the value of $\varsigma w_n$ which is found by $\varsigma w_n > \frac{4.6}{t_s}$ where $t_s$ <4sec, therefore $\varsigma w_n > 1.15$. The below graph shows the plot between the damping ratio and the peak overshoot.

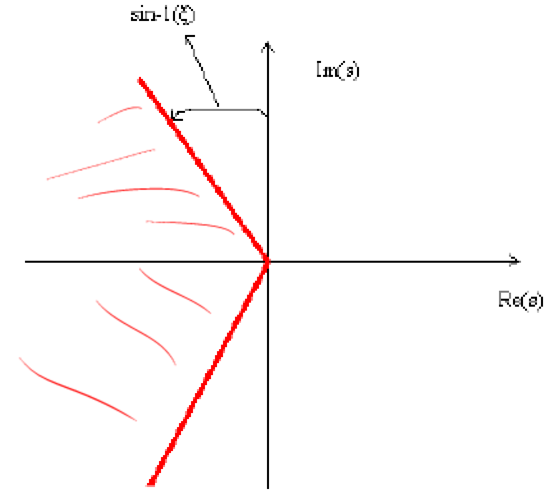


(ADAPTED FROM SATISH NAIR)

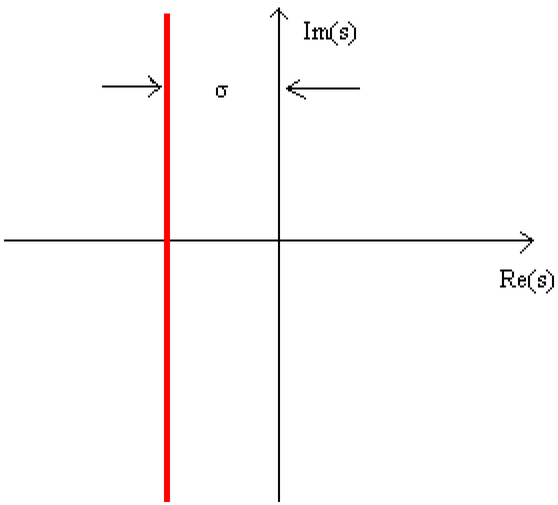


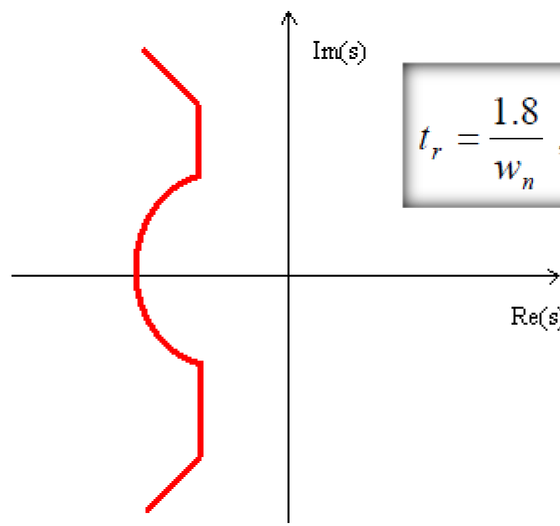

Rise time




Peak overshoot




Settling time




$$t_r = \frac{1.8}{w_n}\,, \quad M_p = e^{-\frac{\varsigma\pi}{\sqrt{1-\varsigma^2}}}\,, \quad t_s = \frac{4.6}{\varsigma w_n}$$

Composite of all three specifications

The above four figures show the graphs of regions in the s-plane delineated by certain transient requirements.

# 3.12 Effect of zeros and extra poles on 'pure' second order system response

**Effect of zeros**

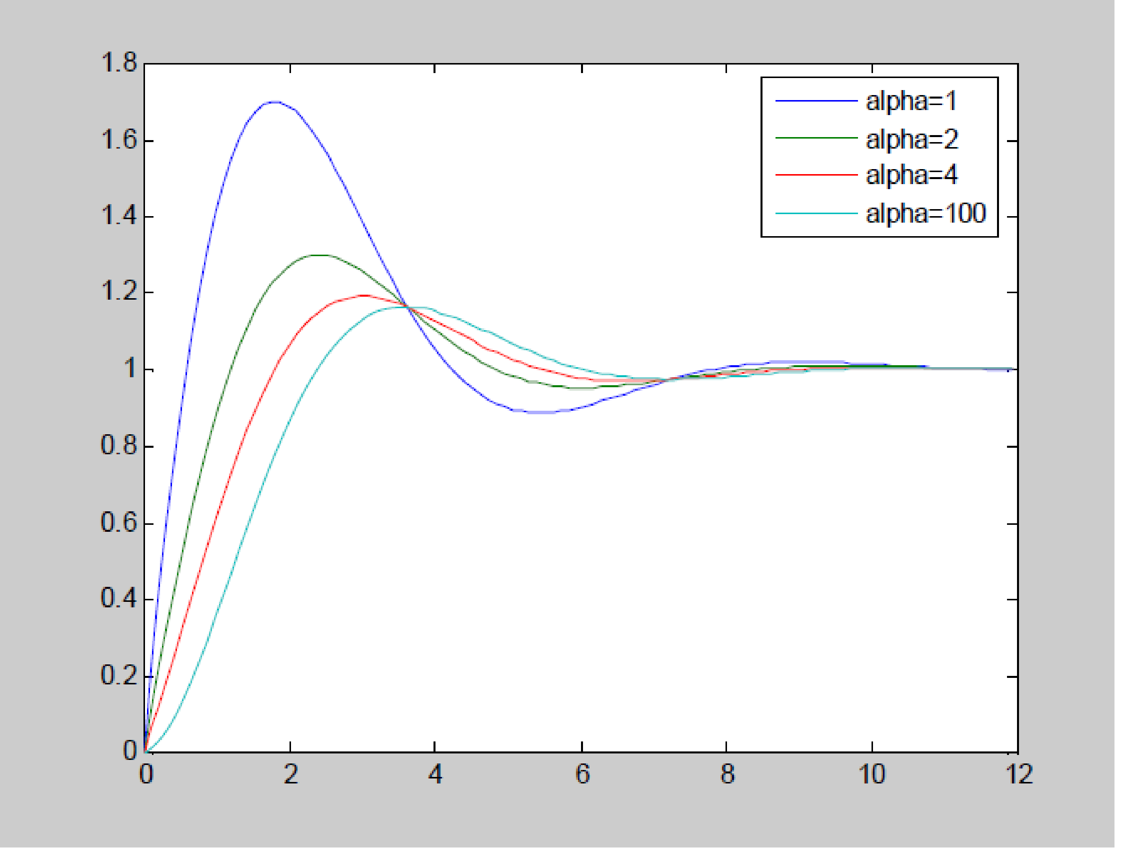
Consider a transfer function $H_1(s)$ as shown below:

$H_1(s) = \dfrac{2}{(s+1)(s+2)} = \dfrac{2}{s+1} - \dfrac{2}{s+2}$ This is a 'pure' second order system, i.e., has two poles only.

Consider the same function with two poles, but now add a zero and normalize it to have the same gain.

In order to show the effect of a zero, placed anywhere in general ($\alpha$ varied in the equation below), on the transient response of the system we consider the following transfer function:

$$H(s) = \frac{\dfrac{s}{\alpha \zeta w_n} + 1}{(s/w_n)^2 + 2\zeta (s/w_n) + 1}$$




**Matlab program**:

```
zeta =0.5;
wn = 1;
alpha = 1;
num = [1/(alpha*zeta*wn) 1];
den = [1/(wn*wn) 2*zeta/wn 1];
sys = tf(num,den);
[y,t]=step(sys);

alpha = 2;
num = [1/(alpha*zeta*wn) 1];
den = [1/(wn*wn) 2*zeta/wn 1];
sys = tf(num,den);
[y1,t1]=step(sys);

alpha = 4;
num = [1/(alpha*zeta*wn) 1];
den = [1/(wn*wn) 2*zeta/wn 1];
sys = tf(num,den);
[y2,t2]=step(sys);

alpha = 100;
num = [1/(alpha*zeta*wn) 1];
den = [1/(wn*wn) 2*zeta/wn 1];
sys = tf(num,den);
[y3,t3]=step(sys);
figure(1);
plot(t,y,t1,y1,t2,y2,t3,y3);
legend('alpha=1','alpha=2','alpha=4','alpha=100');
```

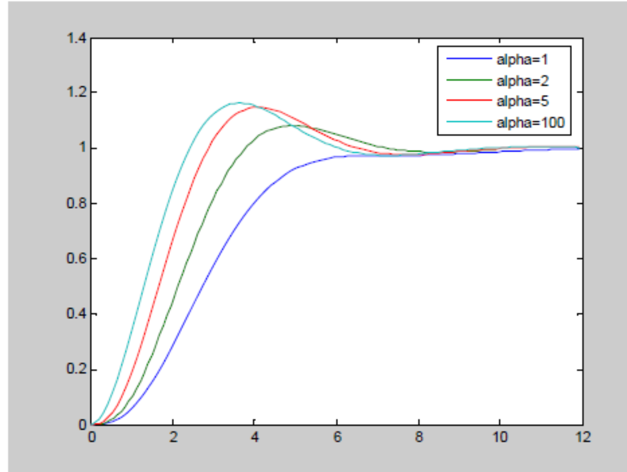
**CONCLUSION**: A zero in the LHP will increase overshoot if the zero is within a factor of 4 of the the real part of the complex poles. A zero in the RHP will depress overshoot (and may cause the step response to start out in the wrong direction.)

## Effect of additional poles

In addition to studying the effects of zeros, it is useful to consider the effects of 'extra' poles on the pure second order step response. Similar to the previous section, consider the pure second order system with an additional pole whose location is controlled again by $\alpha$.

$$H(s)=\frac{1}{(s/\alpha\varsigma w_n+1)\left[(s/\alpha\varsigma w_n)^2+2\varsigma(s/w_n)+1\right]}.$$ Let $\varsigma=0.5$ and $w_n=1$.

An additional pole in the left half-plane will increase the rise time significantly which can be seen in the step response plot below:




**Matlab program.**

```
zeta =0.5;
wn = 1;
alpha = 1;
num = [1];

den=[1/(alpha*zeta*wn^3)((2/alpha)+1)*(1/wn^2)(1/(alpha*zeta)+2*zeta)*(1/wn) 1];
sys = tf(num,den);
[y,t]=step(sys);

alpha = 2;
num = [1];
den=[1/(alpha*zeta*wn^3)((2/alpha)+1)*(1/wn^2)(1/(alpha*zeta)+2*zeta)*(1/wn) 1];
sys = tf(num,den);
[y1,t1]=step(sys);

alpha = 5;
num = [1];
den=[1/(alpha*zeta*wn^3)((2/alpha)+1)*(1/wn^2)(1/(alpha*zeta)+2*zeta)*(1/wn) 1];
sys = tf(num,den);
[y2,t2]=step(sys);

alpha = 100;
num = [1];
den=[1/(alpha*zeta*wn^3)((2/alpha)+1)*(1/wn^2)(1/(alpha*zeta)+2*zeta)*(1/wn) 1];
sys = tf(num,den);
[y3,t3]=step(sys);
figure(1);
plot(t,y,t1,y1,t2,y2,t3,y3);
legend('alpha=1','alpha=2','alpha=5','alpha=100');
```

<u>**CONCLUSION**</u>: An additional pole in the LHP will increase rise time significantly if it is within a factor of 4 of the the real part of the complex poles.

# 3.13 STABILITY

An LTI system is said to be stable if all the roots of the transfer function denominator polynomial have negative real parts (i.e.; they are all in the LHP) and unstable otherwise.

Are the following systems stable (assume a(s) is the denominator of the TF) :

1. $a(s) = s + 3$
2. $a(s) = s^2 + 5s + 6$
3. $a(s) = s^2 + 4s^2 + 2s + 9$

# 3.13 STABILITY

An LTI system is said to be stable if all the roots of the transfer function denominator polynomial have negative real parts (i.e.; they are all in the LHP) and unstable otherwise.

**Routh Hurwitz criterion:** Routh and Hurwitz independently in 19th century developed a method that was particularly useful before the advent of software such as Matlab, but is still useful for determining the ranges of the coefficients of polynomials for stability, especially when the coefficients are in the symbolic form.

Consider the LTI system whose transfer function denominator polynomial leads to the characteristic equation

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0 \qquad \text{Eq 1}$$

Assume that the roots $\{p_i\}$ of the characteristic equation are real or complex, but are distinct. The solutions to the differential equation where characteristic equation is given by Eq 1may be written using the partial-fraction expansion as

$$y(t) = \sum_{i=1}^{n} K_i e^{p_i t} \qquad \text{Eq 2}$$

where $\{p_i\}$ are the roots of the polynomial equation and $\{K_i\}$ depend on the initial conditions and zero locations. If a zero were to cancel a pole in the right half plane for the transfer function ,the corresponding $K_i$ would equal zero in the output, but the unstable transient would appear in some internal variable. The system is stable if and only if (necessary and sufficient condition) every term in Eq 2 goes to zero as $t \rightarrow \infty$

$$e^{p_i t} \rightarrow 0 \qquad \text{for all } p_i \qquad \text{Eq 3}$$

This will happen if all the poles of the system are strictly in the LHP, where

$$\text{Re}\ \{p_i\} < 0 \qquad \text{Eq 4}$$

If any poles are repeated, the response must be changed from that of the Eq 2 by including a polynomial in t in place of $k_i$, but the conclusion remains unchanged. This is called internal stability. Therefore, the stability of the system can be determined by computing the location of the roots of the characteristic equation (poles of the CLTF) and determining whether they are all in the LHP. If the system has any poles in the RHP, it is unstable. Hence the jω axis is the stability boundary between asymptotically stable and unstable response. If the system has non repeated jω axis poles, then it is said to be neutrally stable. For example, a pole at the origin (an integrator) results in a non decaying transient. A pair of complex jω axis poles results in an oscillating response (with constant amplitude). If the system has repeated poles on the jω axis, then it is unstable [as it results in $te^{\pm jw_i t}$ terms in Eq 2]. For example a pair of poles at the origin (double integrator) results in an unbounded response.

# Routh Criterion:

- A necessary (but not sufficient) condition for stability is that all the coefficients of the characteristic polynomial be of the same sign.
- A system is stable if and only if all the elements in the first column of the Routh array are positive.

To determine the Routh array we first arrange the coefficients of the characteristic polynomial in two rows beginnir with the first and the second coefficients and followed by the even numbered and odd numbered coefficients:

| | | | | | |
|---|---|---|---|---|---|
| Row n | $s^n$: | 1 | a2 | a4 | ……. |
| Row n-1 | $s^{n-1}$: | a1 | a3 | a5 | ……. |
| Row n-2 | $s^{n-2}$: | b1 | b2 | b3 | ……. |
| Row n-3 | $s^{n-3}$: | c1 | c2 | c3 | ……. |
| . | | | | | |
| . | | | | | |
| . | | | | | |
| . | | | | | |
| Row 2 | $s^2$: | * | * | | |
| Row 1 | s: | * | | | |
| Row 0 | $s^0$: | * | | | |

We compute the elements from the $(n\text{-}2)^{th}$ and $(n\text{-}3)^{th}$ rows as follows:

$$b_1 = -\frac{\det\begin{bmatrix} 1 & a_2 \\ a_1 & a_3 \end{bmatrix}}{a_1} = \frac{a_1 a_2 - a_3}{a_1} \qquad c_1 = -\frac{\det\begin{bmatrix} a_1 & a_3 \\ b_1 & b_2 \end{bmatrix}}{b_1} = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$b_2 = -\frac{\det\begin{bmatrix} 1 & a_4 \\ a_1 & a_5 \end{bmatrix}}{a_1} = \frac{a_1 a_4 - a_5}{a_1} \qquad c_2 = -\frac{\det\begin{bmatrix} a_1 & a_5 \\ b_1 & b_3 \end{bmatrix}}{b_1} = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

$$b_3 = -\frac{\det\begin{bmatrix} 1 & a_6 \\ a_1 & a_7 \end{bmatrix}}{a_1} = \frac{a_1 a_6 - a_7}{a_1} \qquad c_3 = -\frac{\det\begin{bmatrix} a_1 & a_7 \\ b_1 & b_4 \end{bmatrix}}{b_1} = \frac{b_1 a_7 - a_1 b_4}{b_1}$$

$$d_1 = -\frac{\det\begin{bmatrix} b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}}{c_1} \qquad e_1 = -\frac{\det\begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix}}{d_1}$$

$$d_2 = -\frac{\det\begin{bmatrix} b_1 & b_3 \\ c_1 & c_3 \end{bmatrix}}{c_1} \qquad e_2 = -\frac{\det\begin{bmatrix} c_1 & c_3 \\ d_1 & d_3 \end{bmatrix}}{d_1}$$

**Routh's Test:**

Consider a polynomial $a(s) = s^6 + 4s^5 + 3s^4 + 2s^3 + s^2 + 4s + 4$. It satisfies the necessary condition for stability since all the coefficients $\{a_i\}$ are positive and non-zero. Are any of the roots of this polynomial in the RHP?

The routh array for the above polynomial is

$s^6$: 1    3    1    4

$s^5$: 4    2    4    0

$$s^4:\quad \frac{5}{2} = \frac{4\times 3 - 1\times 2}{4} \qquad 0 = \frac{4\times 1 - 4\times 1}{4} \qquad 4 = \frac{4\times 4 - 1\times 0}{4}$$

$$s^3:\quad 2 = \frac{\frac{5}{2}\times 2 - 4\times 0}{\frac{5}{2}} \qquad -\frac{12}{5} = \frac{\frac{5}{2}\times 4 - 4\times 4}{\frac{5}{2}} \qquad 0$$

$$s^2:\quad 3 = \frac{2\times 0 - \frac{5}{2}\times\left(-\frac{12}{5}\right)}{2} \qquad 4 = \frac{2\times 4 - \left(\frac{5}{2}\times 0\right)}{2}$$

$$s:\quad -\frac{76}{15} = \frac{3\times\left(-\frac{12}{5}\right) - 8}{3} \qquad 0$$

$$s^0:\quad 4 = \frac{-\frac{76}{15}\times 4 - 0}{-\frac{76}{15}}$$

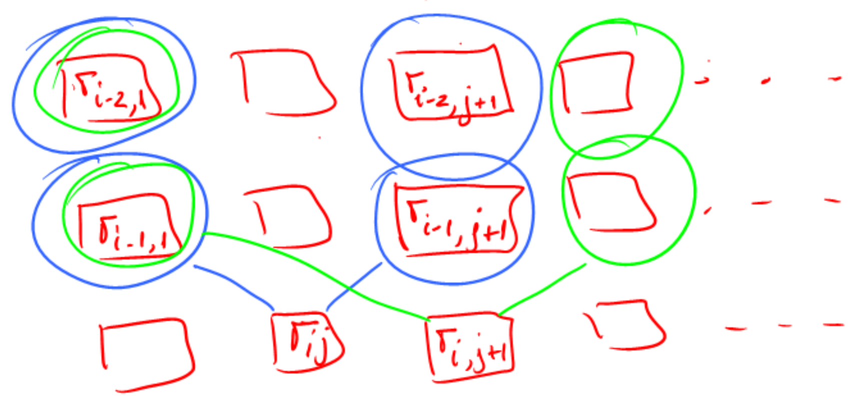
Since the elements of the first column are not all positive we can say that the polynomial has RHP roots. In fact there are two poles in the RHP because there are two sign changes.

FROM THE THIRD ROW DOWN, ANY ELEMENT $r_{ij}$ CAN BE WRITTEN AS :

$$r_{ij} = - \frac{\begin{vmatrix} r_{i-2,1} & r_{i-2,j+1} \\ r_{i-1,1} & r_{i-1,j+1} \end{vmatrix}}{r_{i-1,1}}$$

$i^{th}$ row

$j^{th}$ column



# Special cases of the Routh array

**Case 1:** If only the first element in one of the rows is zero then we replace the zero with a small positive constant $\varepsilon > 0$ and proceed as before. Apply the stability criterion by taking the limit as $\varepsilon \to 0$.

Determine the locations of roots of the polynomial $a(s) = s^5 + 3s^4 + 2s^3 + 6s^2 + 6s + 9$. Form the Routh array.

| | | | | | |
|---|---|---|---|---|---|
| | $s^5$: | 1 | 2 | 6 | |
| | $s^4$: | 3 | 6 | 9 | |
| | $s^3$: | 0 | 3 | 0 | |
| New | $s^3$: | $\varepsilon$ | 3 | 0 | ← Replace zero by $\varepsilon$ |
| | $s^2$: | $\dfrac{2\varepsilon - 3}{\varepsilon}$ | 3 | 0 | |
| | s: | $3 - \dfrac{3\varepsilon^2}{2\varepsilon - 3}$ | 0 | 0 | |
| | $s^0$: | 3 | 0 | | |

In the Routh array, the first element of the 3rd row is zero, and so it is replaced by $\varepsilon$ and the process continued. There are 2 sign changes in the first column in the array which means that there are 2 poles not in the LHP.

**Case 2:** Entire row is zero. This indicates that there are complex conjugate roots that are mirror images of each other with respect to the imaginary axis. If the $i^{th}$ row is zero we form the auxiliary equation from the previous (non-zero) row. We then replace the $i^{th}$ row by the coefficients of the derivative of the auxiliary polynomial, and complete the array. However the roots of the auxiliary polynomial are also roots of the characteristic equation and these must be tested separately. To illustrate this, consider the polynomial $a(s) = s^5 + 5s^4 + 11s^3 + 23s^2 + 28s + 12$.

The Routh array is as shown below:

| | | | | | |
|---|---|---|---|---|---|
| | $s^5$: | 1 | 11 | 28 | |
| | $s^4$: | 5 | 23 | 12 | |
| | $s^3$: | 6.4 | 25.6 | 0 | |
| | $s^2$: | 3 | 12 | | |
| | s: | 0 | 0 | | $\leftarrow a1(s) = 3s2 + 12$ |
| New | s: | 6 | 0 | | $\leftarrow \dfrac{da_1(s)}{ds} = 6s$ |
| | $s^0$: | 12 | | | |

There are no sign changes in the first column. Hence all the roots have negative real parts except for a pair on the imaginary axis. We may deduce this as follows: When we replace the zero in the first column by $\varepsilon > 0$, there are no sign changes. If we let $\varepsilon < 0$, then there are 2 sign changes. Thus if $\varepsilon = 0$, there are 2 poles on the imaginary axis, which are the roots of

$$a(s) = s^2 + 4 = 0$$

or

$$s = \pm j2$$

This agrees with the fact that the actual roots are at -3, $\pm 2j$, -1 and -1.