# 4.5 PID Control

**General**

![PID Control Diagram](image)

**Objective:** Determine how the C.L. roots are related to the control gains $K_p$, $K_I$, and $K_d$ (use SIMULINK to check some with $G(s) = 1/(s+10)$, and different $D(s)$)

<table>
<thead>
<tr>
<th>Control Type</th>
<th>Time Domain Representation</th>
<th>Frequency domain representation of controller</th>
<th>Characteristic equation (denom. of closed loop TF=0), $1 + D(s)G(s) = 0$</th>
<th>Remarks</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>$u(t) = K_p e(t)$</td>
<td>$D(s) = K_p$</td>
<td>$s^2 + a_1 s + a_2 + AK_p = 0$</td>
<td>*1 degree of freedom, and so cannot locate the roots at any location. *Can reduce error magnitude but non-zero steady-state error exists. *Can increase the speed of response but has much larger transient overshoot.</td>
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<tr>
<td>I</td>
<td>$u(t) = K_I \int e(t) dt$</td>
<td>$D(s) = \frac{K_I}{s}$</td>
<td>$s^3 + a_1 s^2 + a_2 s + AK_I = 0$</td>
<td>*Can reduce or eliminate constant steady-state errors at the cost of worse transient response.</td>
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<tr>
<td>PI</td>
<td>$u(t) = \left[ K_p e(t) + K_I \int e(t) dt \right]$</td>
<td>$D(s) = \frac{K_p s + K_I}{s}$</td>
<td>$s^3 + a_1 s^2 + (a_2 + AK_p) s + AK_I = 0$</td>
<td>*2 degrees of freedom and the designer can choose $K_p$ and $K_I$ independently to provide better transient response. *Approximated by lag controller.</td>
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<tr>
<td>D</td>
<td>$u(t) = K_d \dot{e}(t)$</td>
<td>$D(s) = K_d s$</td>
<td>$s^2 + (a_1 + AK_d) s + a_2 = 0$</td>
<td>*Pure derivative feedback is not practical to implement.</td>
</tr>
<tr>
<td>PD</td>
<td>$u(t) = (K_p e(t) + K_d \dot{e}(t))$</td>
<td>$D(s) = (K_p + K_d s)$</td>
<td>$s^2 + (a_1 + AK_d) s + (a_2 + AK_p) = 0$</td>
<td>*Can increase the damping and generally improve the stability of a system. *Approximated by lead controller.</td>
</tr>
<tr>
<td>PID</td>
<td>$u(t) = \left[ K_p e(t) + K_I \int e(t) dt + K_d \dot{e}(t) \right]$</td>
<td>$D(s) = \left( K_p + \frac{K_I}{s} + K_d s \right)$</td>
<td>$s^3 + (a_1 + AK_d) s^2 + (a_2 + AK_p) s + AK_I = 0$</td>
<td>*3 degrees of freedom and can provide an acceptable degree of error reduction simultaneously with acceptable stability and damping.</td>
</tr>
</tbody>
</table>

Other controller types: state feedback, adaptive, robust, stochastic, neural, fuzzy...
Recall: \[ Y = TR + GSW - TV \]
\[ E = SR - GSW + TV \]
\[ (E - R - Y) \]
\[ T = \frac{GD}{1+GD} \]
\[ S = \frac{1}{1+GD} \]
\[ T = 1 - S \]

Open Loop

\[ G = \frac{b}{a} \quad D = \frac{c}{a} \quad T = GD = \frac{bc}{ad} \]

If \( G \) is unstable - i.e., roots (poles) of \( G \) in RHP - one might think that the same roots in \( C \) (zeros) could cancel those poles if unstable (sensitivity!)

Similarly, if \( G \) has poor response due to some zeros of \( b \) in RHP, poles of \( a \) to cancel those zeros would, again, lead to an unstable system.

Sensitivity

\[ S_{Tol} = \frac{STol}{Tol} = \frac{\delta G}{SG} = 1 \]

10% in \( G \) => 10% in \( T_{ol} \)
Feedback 

\[ G = \frac{b}{a}, \quad D = \frac{c}{d}, \quad T = \frac{bc}{ad + bc} = \frac{bc}{ad + bc} \]

Now, if \( G \) is unstable - poles of \( a \) in \( \mathbb{R}NP \).

The exact same zeros in \( b \) will not cancel.

The poles = do now to cancel unstable poles?

By adding \underline{stable} zeros ?!

Ex: \[ G = \frac{b}{a} = \frac{1}{s^2 - 1} = \frac{1}{(s+1)(s-1)} \]

\[ D = \frac{c}{d} = \frac{k(s+1)}{(s+\delta)} \]

\[ T = \frac{bc}{ad + bc} = \frac{k(s+1)}{(s+1)(s-1)(s+\delta) + k(s+1)} \]

\[ T = \frac{k}{(s-1)(s+\delta) + k} = \frac{k}{s^2 + (\delta-1)s + (k-\delta)} \]

\[ \text{Routh criterion} \to \text{choice of } k, \delta \quad (k > \delta > 1) \]
Also,
\[ T_{cl} + \delta T_{cl} = \frac{(G + \delta G)D}{1 + (G + \delta G)D} \]
\[ \delta T_{cl} \approx \frac{dT_{cl}}{dG} \delta G \]

\[ S_{G} \leq \frac{\delta T_{cl}}{T_{cl} \delta G} = \frac{G \delta T_{cl}}{T_{cl} \delta G} \leq \frac{G}{T_{cl}} \frac{dT_{cl}}{dG} = \]

\[ S_{G} = \frac{G}{T_{cl}} \frac{d}{dG} \left( \frac{GD}{1 + GD} \right) = \ldots = \frac{1}{1 + GD} \]

If gain of \(1 + GD = 100 \Rightarrow 10\% \text{ in } G \Rightarrow 0.1\% \text{ in } T_{cl} \)

**Note:** \[ S_{G} = S = \frac{1}{1 + GD} \]
System Type (for tracking the input)

\[ K \mid \text{For } R(t) = t^k \therefore e_{ss} = \lim_{t \to \infty} e(t) = c^t \quad (\text{for } w = v = 0) \]

\[ e_{ss} = \lim_{s \to 0} s \cdot e(s) = c^t \quad \text{if roots } e(t) \in \text{LHP} \]

\[ e_{ss} = \lim_{s \to 0} s \cdot R = \lim_{s \to 0} \frac{s}{1 + GD} = \frac{1}{s^{k+1}} \lim_{s \to 0} \frac{1}{1 + GD} \frac{1}{s^k} \]

\[ \text{if } D = \frac{c}{a} = \frac{c}{s^n} \Rightarrow n \text{ integrators} \]

\[ e_{ss} = \lim_{s \to 0} \frac{1}{1 + GC} \frac{1}{s^k} = \frac{s^n}{s^n + GC} \frac{1}{s^k} \]

If \( GC(0) = Kn = 0 \) System Type = \( n \)

Recall:
\[ n = 2 \quad \text{input? step ramp parabolic} \]

\[ e_{ss} = 0 \quad 0 \quad 0 \quad c^t \]
**System Type** (for regulation - disturbance rejection)

\[ K \]

For \( W(t) = t^r \):
\[ e_{ss} = \lim_{t \to \infty} e(t) = \frac{c t^r}{t^\infty} \]

\[ e(s) = R(s) - Y(s) = -Y(s) \Rightarrow \frac{e(s)}{W(s)} = -\frac{Y(s)}{W(s)} = Tw(s) \]

Taking \( \lim_\infty \) or \( \lim_{s \to 0} \)

\[ y_{ss} = \lim_{s \to 0} Tw(s) \cdot \frac{1}{s^{k+1}} = \lim_{s \to 0} Tw(s) \cdot \frac{1}{s^k} \]

**Ex.**

\[ D = k_p \quad \text{and} \quad D = k_p + \frac{k_I}{s} \]

Type 0

Type 1

\[ y_{ss} = -\frac{b}{A k_p} \]

\[ \text{Yss} = -\frac{b}{A k_I} \]
**Proportional Control**

The control variable $e(t) = K_p e(t)$, where control is proportional to error.

$$\frac{U(s)}{E(s)} = \frac{D_{cl}(s)}{E(s)} = K_p \quad \text{and} \quad \tau(s) = \frac{K_p G(s)}{1 + K_p G(s)}$$

If $G(s) = \frac{b}{a} = \frac{A}{s^2 + a_1 s + a_2}$, then $\tau(s) = \frac{bc}{ad + bc} = \frac{A}{s^2 + a_1 s + a_2 + K_p A}$

System type for tracking is 0:

$$e_{ss} = \frac{1}{a_2 + K_p A}$$

Graph showing system response with $k_p = 6$ and $k_p = 1.5$.
INTEGRAL CONTROL

\[ U(t) = K_I \int_{0}^{t} e(t) dt \]  
Control is the Accumulated Error

\[ \frac{U(s)}{E(s)} = D_{eq}(s) = \frac{K_I}{s} \]  
\[ T(s) = \frac{K_I G(s)}{1 + K_I G(s)} = \frac{K_I G(s)}{s + K_I G(s)} \]

If \( G(s=0) = 1 \)

\[ e_{ss} = \lim_{s \to 0} \frac{s}{s + GK_I} \ast \frac{1}{s^2} \]

System Type for Tracking = 1

\[ e_{ss} = 0 \text{ for } R(s) = \frac{1}{s} \text{ (unit step)} \]

\[ e_{ss} = \frac{1}{K_I} \text{ for } R(s) = \frac{1}{s^2} \text{ (ramp)} \]

Error (t)

\[ e(t) = k_1 \int_{0}^{t} e(\tau) d\tau = k_1 \cdot \text{area} \]

Time (sec)

Area

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\[ Y_{ss} = \lim_{s \to 0} s \cdot T_W(s) \cdot \frac{1}{s^{k+1}} = \lim_{s \to 0} s \cdot \frac{G}{1 + DG} \cdot \frac{1}{s^{k+1}} = \lim_{s \to 0} \frac{Gs}{s + KiG} + \frac{1}{s^k} \]

**System Type for Regulation = 1**

\[ Y_{ss} = 0 \quad \text{for} \quad W(s) = \frac{1}{s} \quad (\text{step}) \]

\[ Y_{ss} = \frac{1}{Ki} \quad \text{for} \quad W(s) = \frac{1}{s^2} \quad (\text{ramp}) \]

\[ U_{ss} = \lim_{s \to 0} s \cdot -DG \cdot \frac{1}{s^{k+1}} = \lim_{s \to 0} s \cdot \frac{-KiG}{s + KiG} \cdot \frac{1}{s^k} \]

\[ U_{ss} = -1 \quad \text{for} \quad W(s) = \frac{1}{s} \quad (\text{step}) \]

\[ U_{ss} = \infty \quad \text{for} \quad W(s) = \frac{1}{s^2} \quad (\text{ramp}) \]
**DERIVATIVE CONTROL**

\[ U(t) = K_D \dot{e}(t) \quad \text{CONTROL IS THE RATE OF THE ERROR} \]

\[ \frac{U(s)}{E(s)} = D_{ci}(s) = K_D s \quad \text{T}(s) = \frac{K_D s G(s)}{1 + K_D s G(s)} = \frac{K_D s G(s)}{1 + K_D s G(s)} \]

\[ G(s) = \frac{b}{a} = \frac{A}{s^2 + a_1 s + a_2} \]

\[ T(s) = \frac{bc}{ad + bc} = \frac{A K_D s}{s^2 (a_1 + K_D A) s + a_2} \]
\[ u(t) = K_p e(t) + K_i \int_0^t e(t) \, dt + K_D \frac{d}{dt} e(t) \]

\[ u(s) = \left( K_p + \frac{K_i}{s} + K_D s \right) e(s) \]

\[ G(s) = s^2 + a_1 s + a_2 \]