LECTURE 30

GUIDELINES TO DETERMINING ROOT LOCUS

DEF 1 RL IS THE SET OF VALUES OF S FOR WHICH $1+KL(s)=0$ AS $K=0\to\infty$

THE ROOTS OF THIS CE, $1+KL(s)=0$, ARE THE POLES OF A CLTF OF A CONTROL SYSTEM

DEF 2 A POINT s_0 IN THE RL SATISFIES

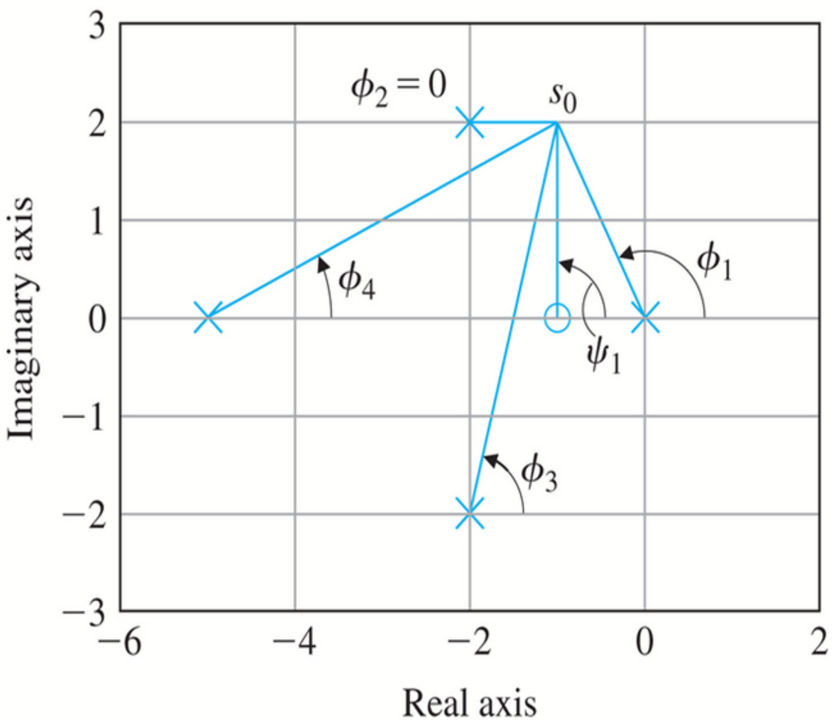
$$L(s) = -1/K \Rightarrow \text{FOR } K \geq 0,\ L(s_0)$$

IS A NEGATIVE REAL NUMBER

$$\Rightarrow \angle L(s_0) = 180^\circ + 360^\circ(\ell-1) \quad \forall \ell \in \mathbb{N}^+$$

(ADAPTED FROM SATISH NAIR)

RECALL $\angle L(s_0) = \sum \angle(s_0 - z_i) - \sum \angle(s_0 - p_i) = \sum \varphi_i - \sum \phi_i$



$\rightarrow \quad L(s) = \dfrac{s+1}{s(s+5)\left[(s+2)^2+4\right]}$

$\Rightarrow \quad z = -1$

$p = 0, -5, -2 \pm 2j$

IS $s_0 = -1 + 2j$ A ROOT OF $1 + K\,L(s) = 0$ FOR SOME $K = 0 \rightarrow \infty$?

$$\angle L(s_0) = 180^\circ + 360(\ell - 1) = \angle \frac{(-1+2j+1)}{(-1+2j)(-1+2j+5)\left[(-1+2j+2)^2+4\right]}$$

$$= \varphi_1 - (\phi_1 + \phi_2 + \phi_3 + \phi_4) = 90^\circ - 116.6^\circ - 0^\circ - 76^\circ - 26.6^\circ = -129.2^\circ$$

NO!

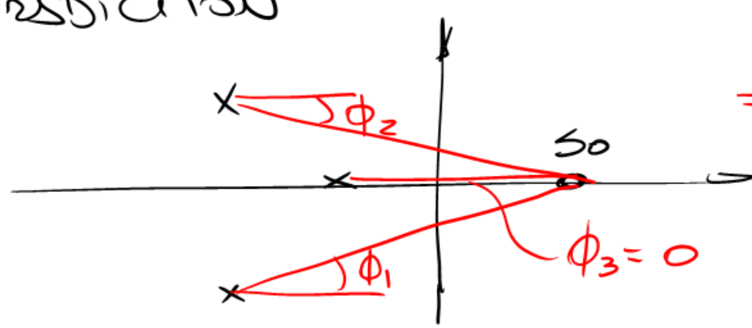
RULES FOR DETERMINING POSITIVE ROOT LOCUS
(i.e. $K \geq 0$)

R1 BRANCHES OF THE RL START @ THE n POLES OF $L(s)$ AND MOVE TOWARDS ITS m ZEROS

- RECALL $a(s) + K\,b(s) = 0$
  - $K = 0 \Rightarrow a(s) = 0$
  - $K = \infty \Rightarrow b(s) = 0$
  - $L(s) = -1/K$

R2 THE LOCI ARE ON THE REAL AXIS TO THE LEFT OF AN ODD # OF POLES + ZEROS

- CONTRADICTION



$\Rightarrow \angle L(s_0) = 0^\circ \neq 180 - 360(\ell - 1)$

R3 FOR LARGE s & K, n−m BRANCHES OF THE LOCI ARE ASYMPTOTIC TO LINES @ ANGLES ϕ_ℓ RADIATING OUT OF $s_0 = \alpha$

WHERE

$$\alpha = \frac{\sum p_i - \sum z_i}{n-m} \quad \& \quad \phi_\ell = \frac{180^\circ + 360^\circ(\ell-1)}{n-m}$$

– AGAIN, AS $K \to \infty$, $L(s) = -1/K \Rightarrow 0$

CASE 1: RULE 1

CASE 2: $s \to \infty$ & $1 + KL(s) = 1 + K\frac{b(s)}{a(s)}$

$|a(s)| > |b(s)| \Rightarrow L(s) \to 0$

$$1 + KL(s) \cong 1 + K\frac{1}{(s-\alpha)^{n-m}} = 0$$

Ex $L(s) = \dfrac{1}{s[(s+4)^2+16]} = \dfrac{1}{s\left(s+(-4+4j)\right)\left(s+(-4-4j)\right)}$ $m = 0$, $n = 3$

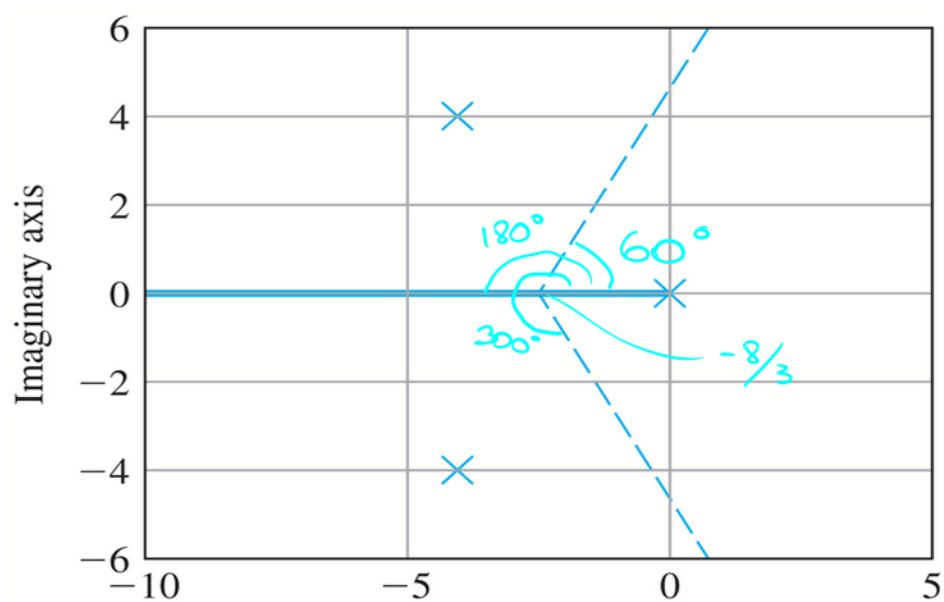
$\Rightarrow \alpha = \dfrac{\sum p_i - \sum z_i}{n-m} = \dfrac{-4+4j-4-4j-0}{3}$ ALWAYS THE CASE (ONLY Re matters)

$\alpha = \dfrac{-8}{3}$

$\phi_l = \dfrac{180 + 360(l-1)}{3}$

$= 60 + 120(l-1)$

$= \begin{cases} 60^\circ \\ 180^\circ \\ 300^\circ \ (-60^\circ) \end{cases}$



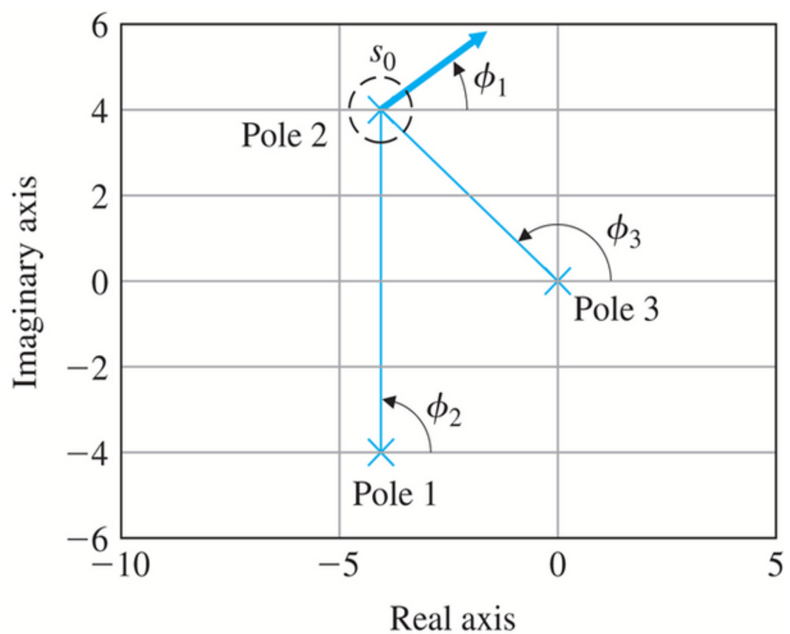
R4

THE ANGLES OF {DEPARTURE / ARRIVAL} OF A BRANCH OF THE LOCUS FROM A {POLE / ZERO} ARE GIVEN BY

$$q \cdot \phi_{DEP} = \sum \varphi_i - \sum_{i \neq dep} \phi_i - 180^\circ - 360^\circ(\ell - 1)$$

$$q \cdot \phi_{ARR} = \sum \phi_i - \sum_{i \neq arr} \varphi_i + 180^\circ + 360^\circ(\ell - 1)$$

q: MULTIPLICITY OF THE POLE/ZERO



IMAGINE
$s_0 \approx$ POLE (DEPARTURE)
&
$s_0 \approx$ ZERO (ARRIVAL)

$\phi_{DEP / ARR}$ = "WHAT IS LEFT"

FOR THE EXAMPLE ABOVE $\left(L(s) = \frac{1}{s(s+(-4+4j))(s+(-4-4j))} \right)$

$$\phi_{DEP_1} = -\phi_2 - \phi_3 - 180^\circ - 360(\ell-1) = -90^\circ - 135^\circ - 180^\circ$$

$$\phi_{DEP_1} = -45^\circ \quad \text{(POLE 1)}$$

SIMILARLY,

$$\phi_{DEP_2} = 90^\circ + 135^\circ - 180^\circ$$

$$\phi_{DEP_2} = 45^\circ \quad \text{(POLE 2)}$$

AND

$$\phi_{DEP_3} = 135 - 135 - 180^\circ$$

$$\phi_{DEP_3} = 180^\circ \quad \text{(POLE 3)}$$

R5

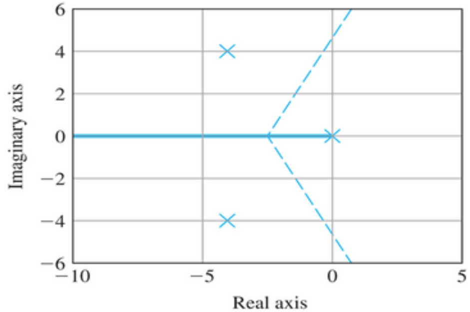
THE LOCUS CAN HAVE MULTIPLE ROOTS AT POINTS ON THE LOCUS ⟹ BRANCHES WILL APPROACH A POINT OF q ROOTS AT ANGLES SEPARATED BY

$$\frac{180° + 360°(l-1)}{q}$$

Ex

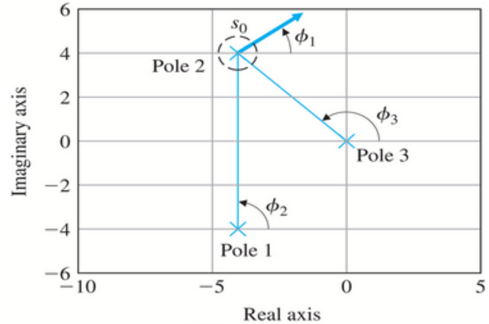
$$L(s) = \frac{1}{s[(s+4)^2+16]} = \frac{1}{s(s+(-4+4j))(s+(-4-4j))} \qquad m = 0 \quad n = 3$$

R3



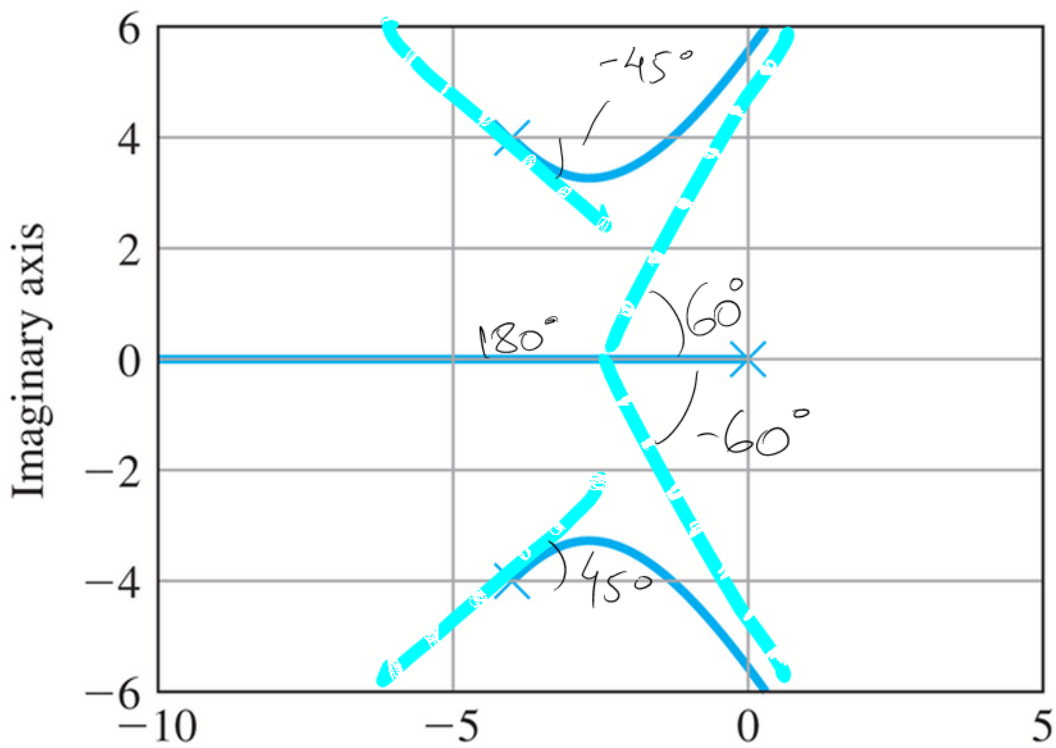
6
4
2
0
−2
−4
−6
−10
−5
0
5
Imaginary axis
Real axis

R4



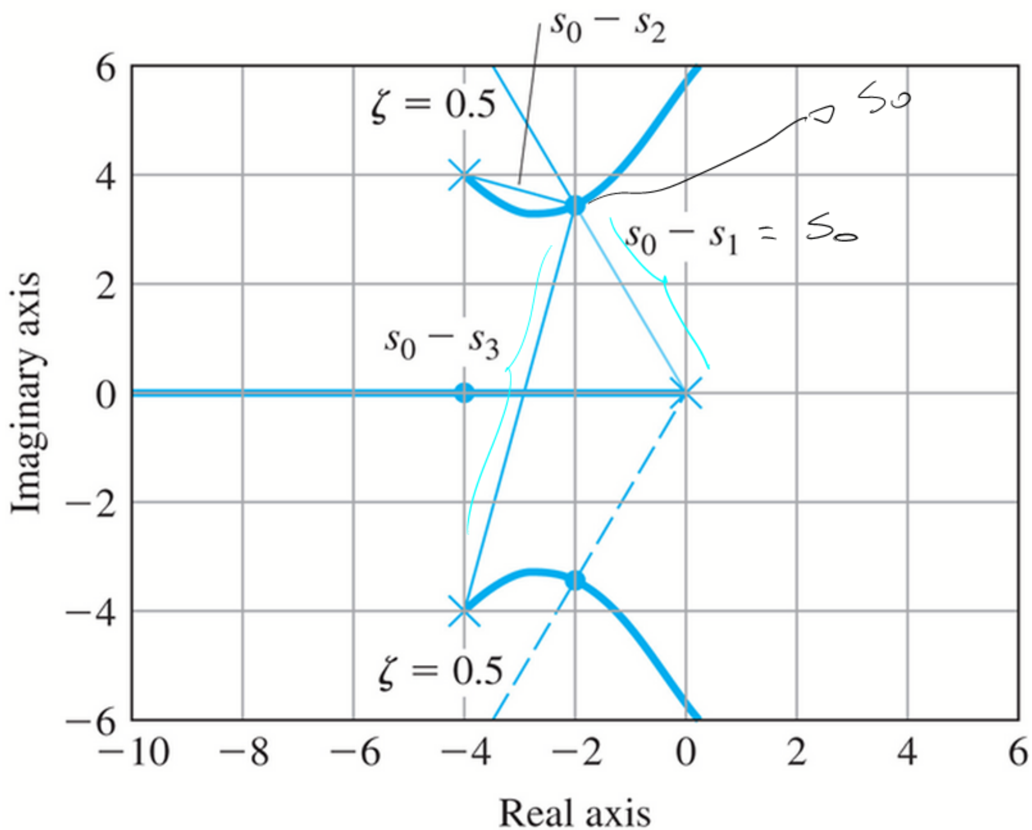
s_0
ϕ_1
Pole 2
ϕ_3
Pole 3
ϕ_2
Pole 1
Imaginary axis
Real axis
6
4
2
0
−2
−4
−6
−10
−5
0
5

R5



-45°
60°
180°
-60°
45°
6
4
2
0
−2
−4
−6
−10
−5
0
5
Imaginary axis
Real axis

Ex Design K such that damping ratio = 0.5



$$K = -\frac{1}{L(s)}$$

$$K = \frac{1}{|L(s_0)|}$$

$$= \frac{1}{|s_0(s_0-s_2)(s_0-s_3)|}$$

$$= |s_0||s_0-s_2||s_0-s_3|$$

$$= 4 * 2.1 * 7.7 \quad \text{(GRAPHICALLY)}$$

K = 65