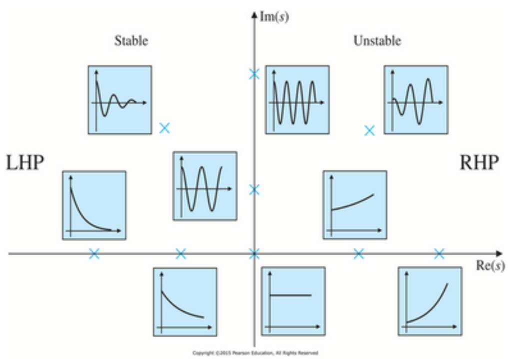
Lecture

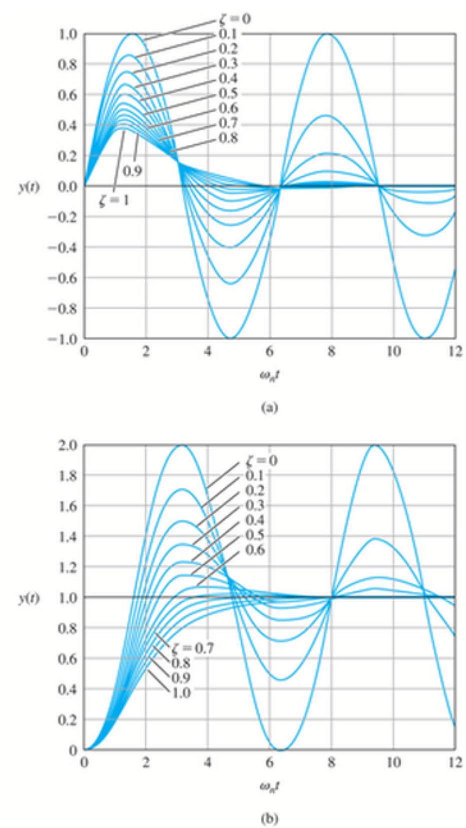
Recall

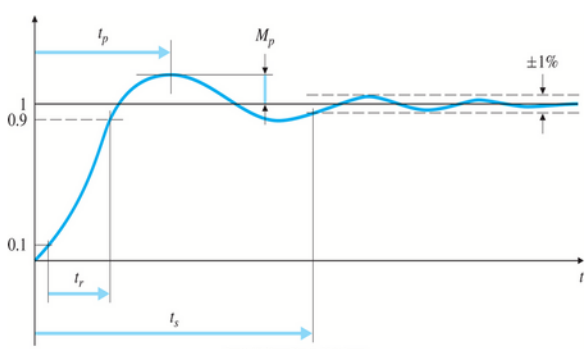
Chapter 1: Feedback - What & Why?

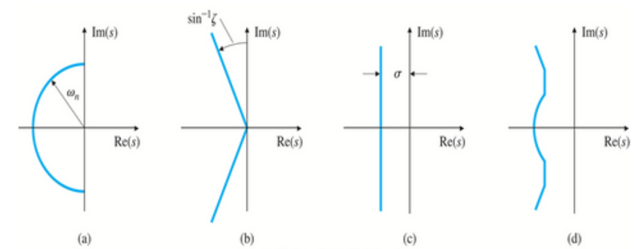
Chapter 2: Models - Block Diagram

Chapter 3: Response - Transfer Function
DEq & Laplace
Impulse, Step . . .
Pole Location









CHAPTER 4 : ANALYSIS - STABILITY

SS ERROR

SYSTEM TYPE

FOR { TRACKING / REGULATION

TYPES OF CONTROLLERS

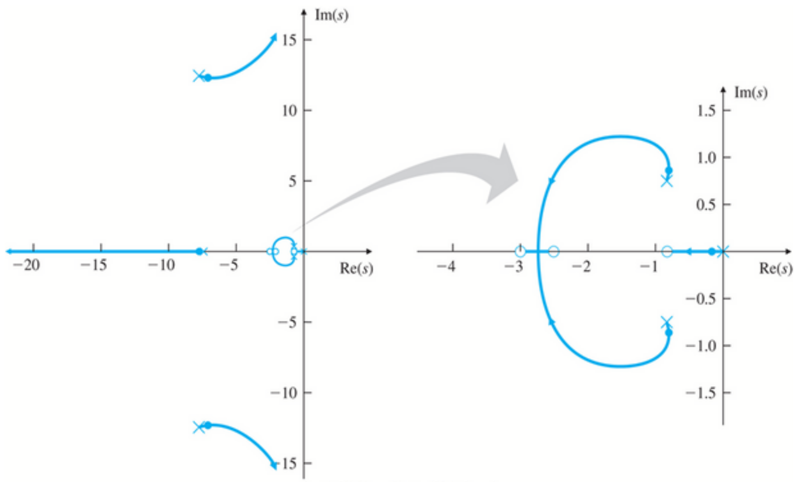
P.I.D.

CHAPTER 5 : DESIGN W/ RL - STABILITY

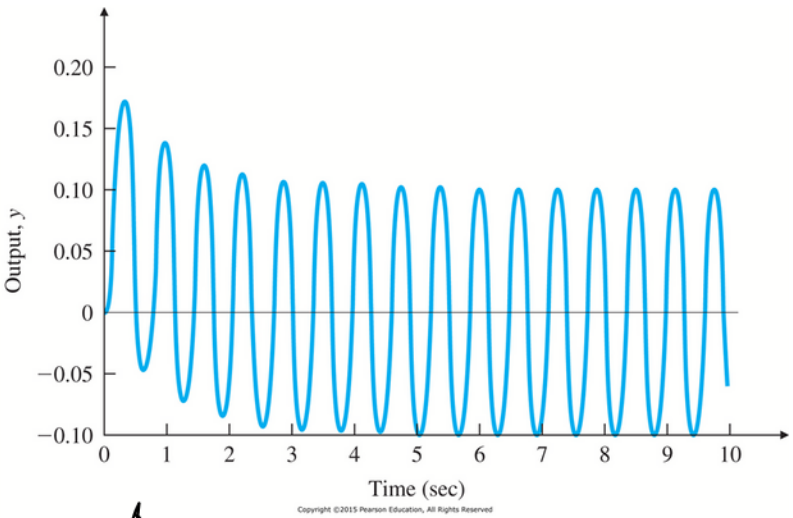
SENSITIVITY

LAG/LEAD/NOTCH

COMPENSATION



Frequency-Response Design



$$\frac{Y(s)}{U(s)} = G(s)$$

if $u(t) = A\sin(\omega_0 t)\,1(t)$

$$U(s) = \frac{A\omega_0}{s^2+\omega_0^2}$$

$$Y(s) = G(s)\,\frac{A\omega_0}{s^2+\omega_0^2}$$

partial fraction expansion

$$Y(s) = \frac{\alpha_1}{s-p_1} + \frac{\alpha_2}{s-p_2} + \cdots + \frac{\alpha_0}{s+j\omega_0} + \frac{\alpha_0^*}{s-j\omega_0}$$

$$y(t) = \underbrace{\alpha_1 e^{p_1 t} + \alpha_2 e^{p_2 t} + \cdots}_{\text{decay } p_i \in Re^-} + 2|\alpha_0|\cos(\omega_0 t + \phi)$$

ϕ

$$y(t) \longrightarrow \underbrace{2|\alpha_0|}_{AM} \cos(\omega_0 t + \phi_0)$$

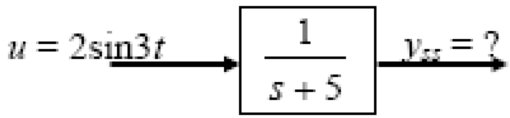
$\phi_0 \longrightarrow \tan^{-1}\left[\frac{Im(\alpha_0)}{Re(\alpha_0)}\right]$

$AM \longrightarrow |G(j\omega_0)|$

$$\angle G(j\omega_0) = \tan^{-1}\left[\frac{Im(G(j\omega_0))}{Re(G(j\omega_0))}\right]$$

$$\boxed{G(j\omega_0) = M e^{j\phi}}$$

Example of usage of Bode plots to determine response to a particular input:



Solution:

1) Substitute $s = i\omega$ into $\frac{1}{s+5}$, and then put ω=3 rad/sec, to get $\frac{1}{3i+5}$.

2) $M = \left|\frac{1}{3i+5}\right| = 0.1715$, $\phi = \arg\left(\frac{1}{3i+5}\right) = -0.5405$ rad

3) $y_{ss} = 2 \times 0.1715 \sin(3t - 0.5405) = 0.343 \sin(3t - 0.5405)$

How would y change if ω changed?

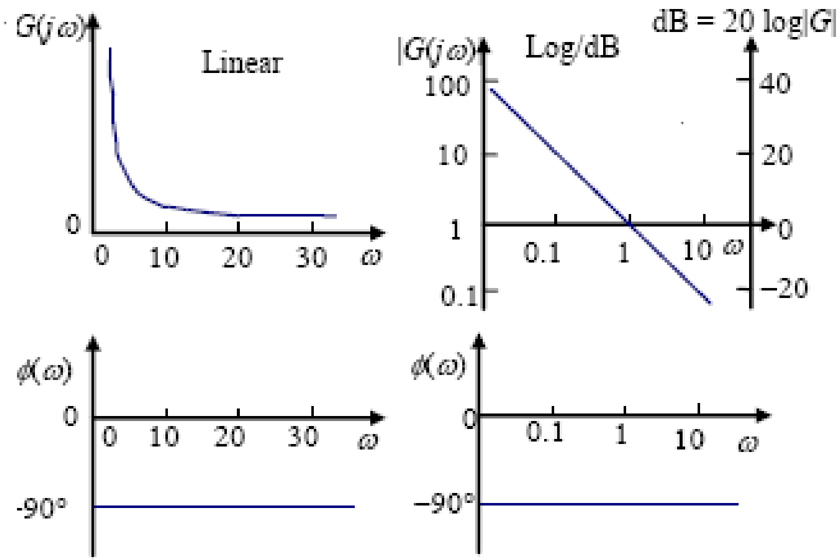
Bode Plot

6.2. BODE PLOTS FOR SIMPLE CASES

$G(s)=\frac{1}{s}\Rightarrow G(j\omega)=\frac{1}{j\omega}$ ~ $j\omega^{-1}$ (*)

Magnitude: $|G(j\omega)|=\frac{1}{\omega}$, Phase: $\angle G(j\omega)=-90^\circ$

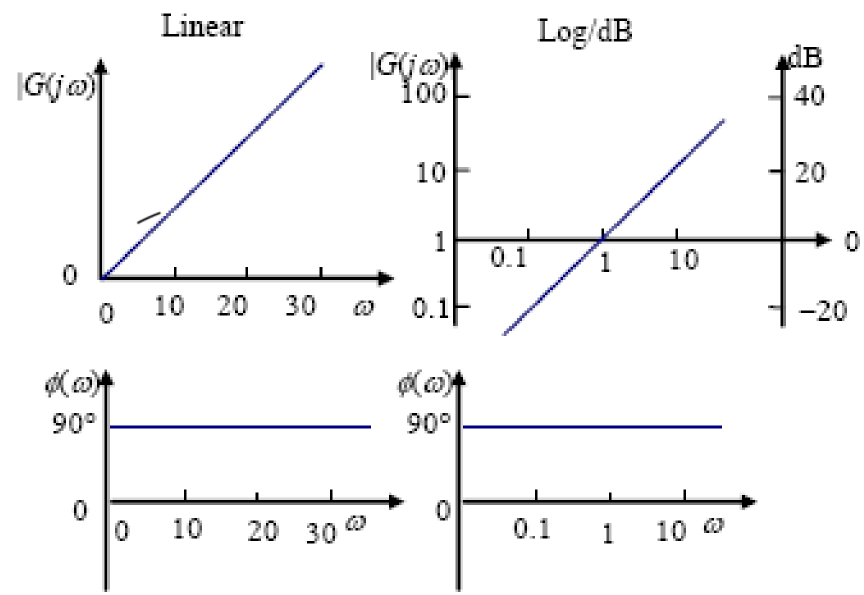
dB: $20\log\left(\frac{1}{\omega}\right)=-20\log\omega$, $\phi(\omega)=-90^\circ$



$G(s)=s\Rightarrow G(j\omega)=j\omega$

Magnitude: $|G(j\omega)|=\omega$, Phase: $\angle G(j\omega)=90^\circ$

dB: $20\log(\omega)=20\log\omega$, $\phi(\omega)=90^\circ$



* in general, for $G(s)=s^n \mid n\in\mathbb{N}$

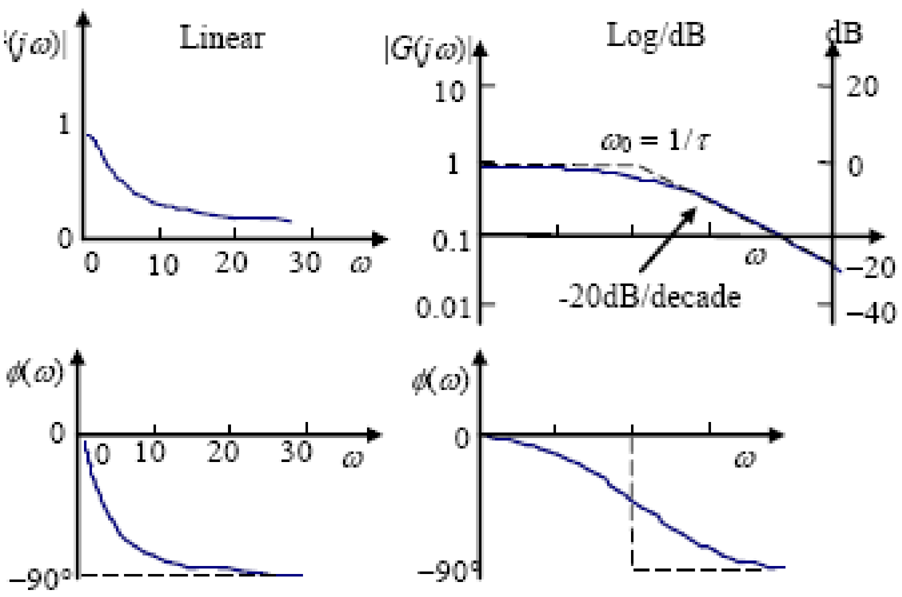
$|G(j\omega)|=\omega^n$ < $\omega, \omega^2, \ldots$; $\frac{1}{\omega}, \frac{1}{\omega^2} \ldots$

$\angle G(j\omega)=n90^\circ$ < $90, 180\ldots$; $-90, -180\ldots$

$$G(s)=\frac{1}{\tau s+1}\Rightarrow G(j\omega)=\frac{1}{\tau j\omega+1}$$

Magnitude: $|G(j\omega)|=\frac{1}{\sqrt{\tau^2\omega^2+1}}$, $\phi(\omega)=-\tan^{-1}\omega\tau$

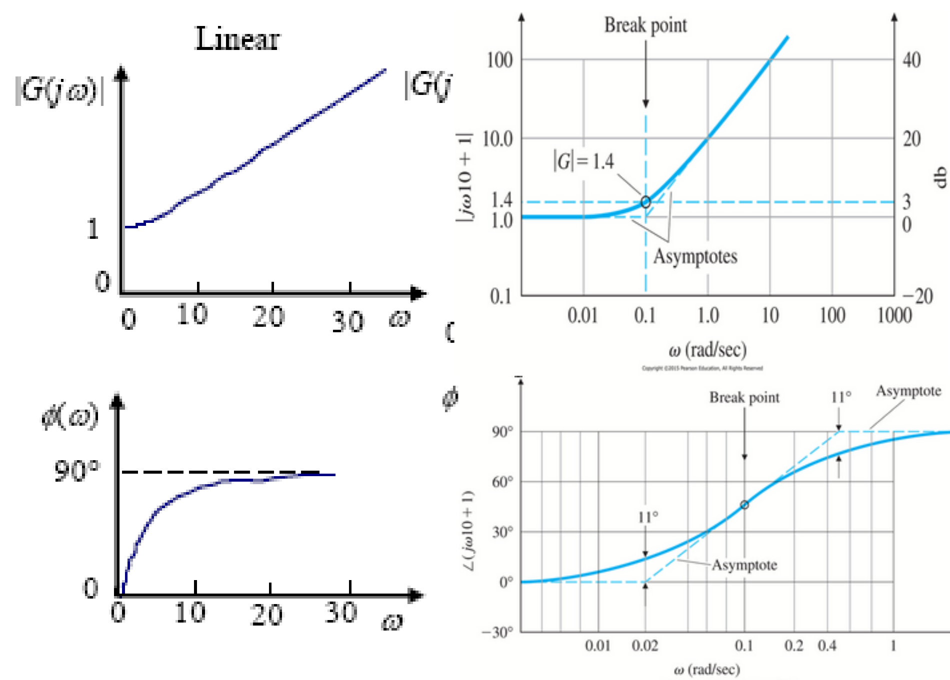
dB: $-20\log\sqrt{\tau^2\omega^2+1}=\begin{cases}0 & \omega\tau\ll 1\\ -20\log\tau\omega & \omega\tau\gg 1\end{cases}$,



$$G(s)=\tau s+1\Rightarrow G(j\omega)=\tau j\omega+1$$

Magnitude: $|G(j\omega)|=\sqrt{\tau^2\omega^2+1}$, $\phi(\omega)=\tan^{-1}\omega\tau$

dB: $20\log\sqrt{\tau^2\omega^2+1}=\begin{cases}0 & \omega\tau\ll 1\\ 20\log\tau\omega & \omega\tau\gg 1\end{cases}$,



breakpoint $\Rightarrow \omega = 1/\tau$

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

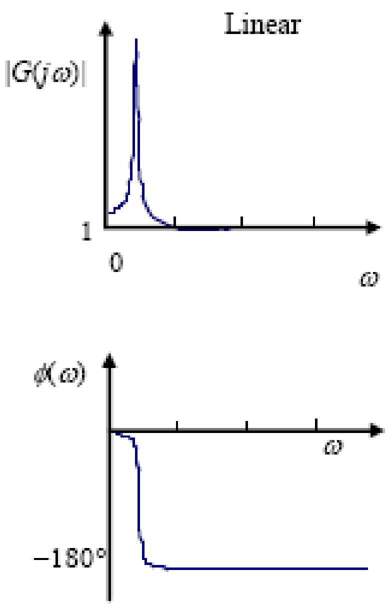
$$\Rightarrow G(j\omega)=\frac{1}{(j\omega/\omega_n)^2+2\zeta(j\omega/\omega_n)+1}$$

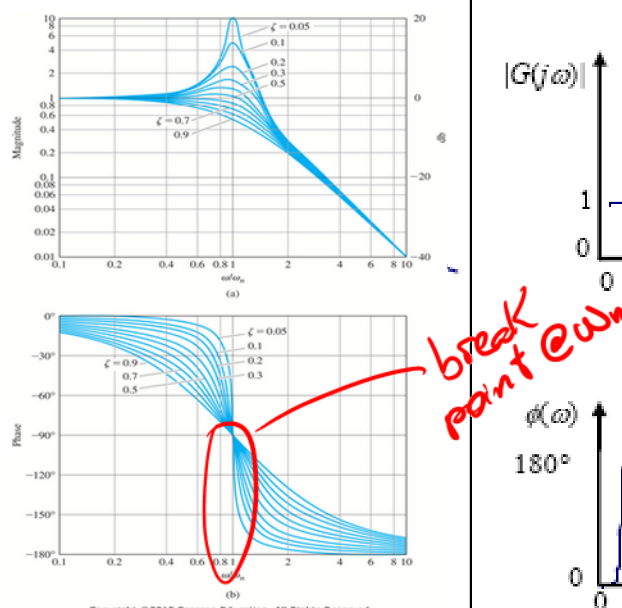
Magnitude: $$|G(j\omega)|=\frac{1}{\sqrt{(1-\omega^2/\omega_n^2)^2+4\zeta^2\omega^2/\omega_n^2}},$$

$$\phi(\omega)=-\tan^{-1}\frac{1-\omega^2/\omega_n^2}{2\zeta\,\omega/\omega_n}$$

dB: $$-20\log|G(j\omega)|=\begin{cases}0 & \omega\ll\omega_n\\ -20\log 2\zeta & \omega=\omega_n\\ -40\log\omega/\omega_n & \omega\gg\omega_n\end{cases},$$

The plot below is for $\zeta = 0.1$ case is shown below:





$$G(s)=(s/\omega_n)^2+2\zeta\, s/\omega_n+1$$

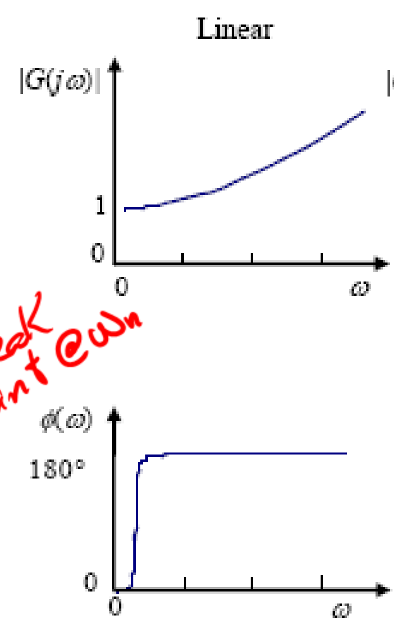
$$\Rightarrow G(j\omega)=(j\omega/\omega_n)^2+2\zeta(j\omega/\omega_n)+1$$

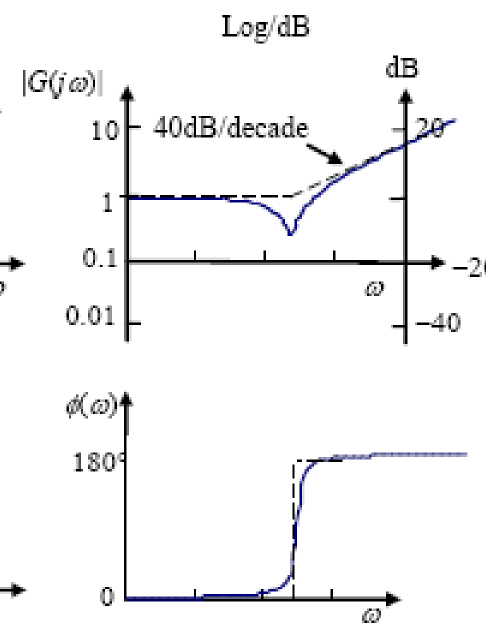
Magnitude: $$|G(j\omega)|=\sqrt{(1-\omega^2/\omega_n^2)^2+4\zeta^2\omega^2/\omega_n^2},$$

$$\phi(\omega)=\tan^{-1}\frac{2\zeta\,\omega/\omega_n}{1-\omega^2/\omega_n^2}$$

dB: $$-20\log|G(j\omega)|=\begin{cases}0 & \omega\ll\omega_n\\ 20\log 2\zeta & \omega=\omega_n\\ 40\log\omega/\omega_n & \omega\gg\omega_n\end{cases},$$

The plot below is for $\zeta = 0.1$ case is shown below:





You can now use the ideas from these 'component' plots, to generate plots for complex transfer functions!
The rationale for this is given below:

Consider $$G(j\omega) = K\frac{(j\omega\tau_1+1)(j\omega\tau_2+1)\cdots}{(j\omega)^n(j\omega\tau_a+1)(j\omega\tau_b+1)\cdots\left(\left(\frac{j\omega}{\omega_n}\right)^2+2\zeta\left(\frac{j\omega}{\omega_n}\right)+1\right)}$$

Taking 20*log10 of both sides (to convert to db), we can convert multiplications and divisions of individual TF terms to additions and subtractions!

$$20\log|G(j\omega)| = 20\log K + 20\log|j\omega\tau_1+1| + 20\log|j\omega\tau_2+1| + \ldots\ldots\ldots\ldots$$

$$- 20{*}n{*}\log\omega - 20\log|j\omega\tau_a+1| - 20\log|j\omega\tau_b+1| - \ldots\ldots - 20\log|(j\omega/\omega_n)2+2\zeta(j\omega/\omega_n)+1)|$$

This will simplify plotting considerably, since all we have to do is add/subtract each of the plots above, as appropriate. The procedure to do this is provided first, followed by an example.

NOTE: You do not take log of the phase plot – it is always linear on the y-axis, but with x-axis on log scale.

BODE PLOTTING PROCEDURE

Step 1: Substitute $s = j\omega$ in the transfer function $G(s)$, to get the sinusoidal transfer function G(jω).

Step 2: Put the system into the form

$$G(j\omega) = K \frac{(j\omega\tau_1 + 1)(j\omega\tau_2 + 1)\cdots}{(j\omega)^n (j\omega\tau_a + 1)(j\omega\tau_b + 1)\cdots\left(\left(\frac{j\omega}{\omega_n}\right)^2 + 2\varsigma\left(\frac{j\omega}{\omega_n}\right) + 1\right)}$$

Step 3: Set the first ω axis value at least 1 decade below lowest break point (p_i or z_i; note = $1/\tau_i$) and the last ω axis value at least 1 decade above the highest break point, making sure to use powers of 10 (i.e., 0.001, 0.01, 0.1, 1, 10, 100 etc.). The break points are z_i for the numerator terms, and p_i for the denominator terms, both of which are calculated as = $1/\tau_i$, with the correct value for i. For a quadratic term, the break point is ω_n.

Step 4: Mark the break points on the ω axis

Step 5: Calculate the low frequency magnitude. For systems with no free poles or zeros, this is roughly the $G(0)$ (DC Gain). Otherwise, if the system has free poles or zeros, a magnitude calculation will have to be made at the first ω axis value from Step 3 or you can draw the low frequency magnitude asymptote through the point K at ω=1 with a slope of n x 20 dB/decade. This is the start of the Bode magnitude plot.

Step 6: Draw the magnitude plot from left to right with a slope of 20 dB / dec. × (#zeros - #poles below current ω value). Thus, the slope will change every time ω approaches a new break point.

IMPROVEMENTS: Increase the asymptote value by a factor of 1.4 (+3 dB) at first-order numerator break points, and decrease it by a factor of 0.707 (-3 dB) at first order denominator break points. At second-order break points, sketch in the resonant peak (or valley) using the relation $|G(j\omega)| = 1/2\varsigma$ at the break point.

Step 7: Evaluate low and high frequency phase angles as follows:

$\angle G(j\omega_{low}) = 90 \times$ (# of free zeros - # of free poles)

$\angle G(j\omega_{high}) = 90 \times$ (# of total zeros - # of total poles)

The phase range will be at least as large as this.

Step 8: Mark the break points and their effective ranges (±1 decade) on the ω axis.

Step 9: Beginning with the low frequency phase from Step 7, draw the phase plot from left to right with a slope of: 45 deg/dec. $\times$ (# of effective zeros - # of effective poles at the current ω value)

If this is done correctly, the high frequency phase portion will have the same value as calculated in Step 7 and have a slope of zero.

EXAMPLE

Draw the Bode log-magnitude and phase plots of $G(s)$ for the unity feedback system where

$$G(s)=\frac{s+3}{(s+0.5)(s^2+20s+400)}$$

Step 1. Substitute $s = j\omega$ in the transfer function $G(s)$.

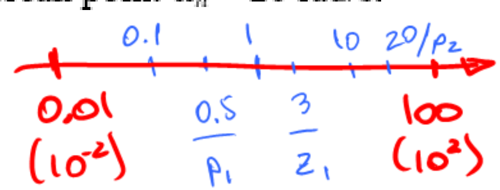
Step 2. Normalize $G(j\omega)$

$$G(j\omega)==(3/200)\frac{(j\omega*0.333+1)}{(j\omega*2.0+1)\left(\left(\frac{j\omega}{20}\right)^2+\left(\frac{j\omega}{20}\right)+1\right)}$$

Step 3. Note that there are three break points: $p_1 = 1/2 = 0.5$, $z_1 = 1/0.33 = 3$ and $\omega_n = 20$.

Therefore, set the first ω axis at 0.01 which is two decade below the lowest break point $p_1 = 0.5$. Then set the last ω axis point at 100 which is about a decade above the highest break point $\omega_n = 20$ rad/s.

Step 4. Mark the break points for the magnitude plot on the ω axis.



Step 5. Calculate the low frequency magnitude.

The low frequency value for $G(s)$, found by letting $s = 0$, is $3/200 = 0.015$, or -36.48dB.

Step 6. Draw the magnitude plot - See Figure 1 on the next page.

Note: The correction to the log-magnitude curve due to the underdamped second order term can be found by plotting a point $20*\log(2\zeta)$ above the asymptotes at the natural frequency. Since $\zeta = 0.5$ for the second order term in the denominator of $G(s)$, the correction is 7.96 dB.

SLOPE

		$p_i = 0.5$	$z_i = 3$	$\omega_n = 20$
Frequency (rad/s)	0.01	0.5	3	5
p_i=0.5	0	-20	-20	-20
z_i=3	0	0	20	20
ω_n=20	0	0	0	-40
Total slope (dB/dec)	0	-20	0	-40

Step 7. Evaluate low and high frequency phase angles $\angle G(j\omega_{low}) = 0°$ and $\angle G(j\omega_{high}) = 90°(1\text{-}3) = \text{-}180°$

Step 8. Mark the break points and their effective ranges for the phase angle plot on the ω axis.

ANGLE

	Start $p_i = 0.5$	Start $z_i = 3$	Start $\omega_n = 20$	End $p_i = 0.5$	End $z_i = 3$	End $\omega_n = 20$
Frequency (rad/s)	0.2	0.3	0.5	20	30	50
$p_i = 0.5$	-45	-45	-45	0	0	0
$z_1 = 3$		45	45	45	0	0
$\omega_n = 20$			-90	-90	-90	0
Total slope (deg/dec)	-45	0	-90	-45	-90	0

Step 9. Draw the phase plot - See Figure 1 on the next page.

Note: The MATLAB commands below can also give you the Bode plots:

```
Num=[1 3];
Den=conv([1 0.5],[1 20 400]);
bode(Num,Den)
```

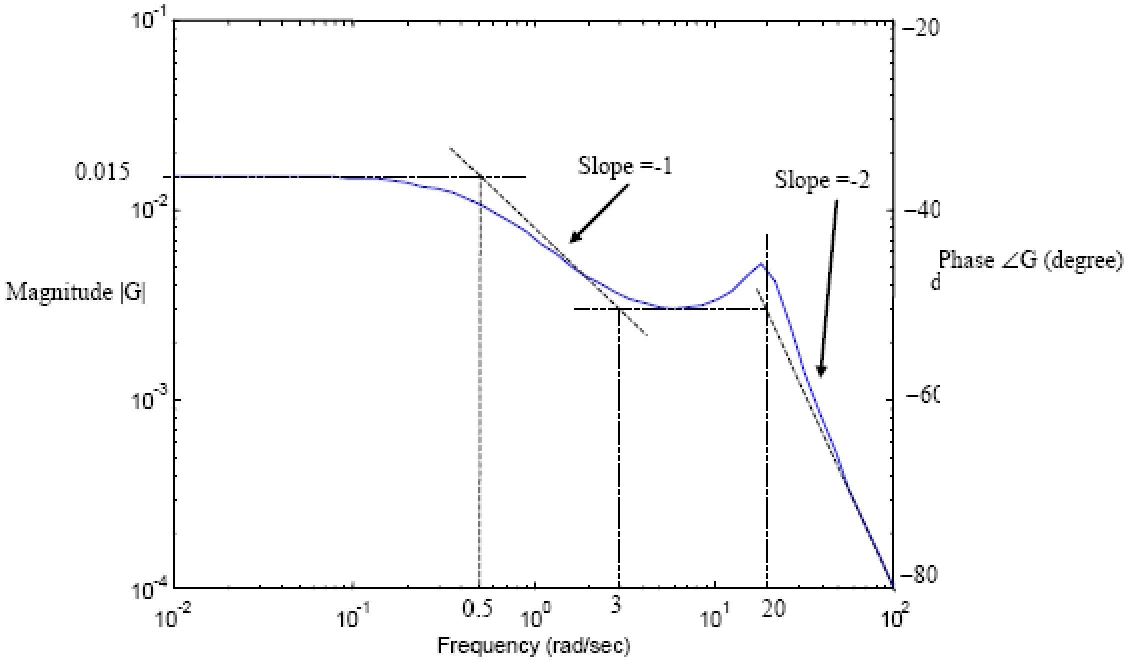


Figure 1. Magnitude plot

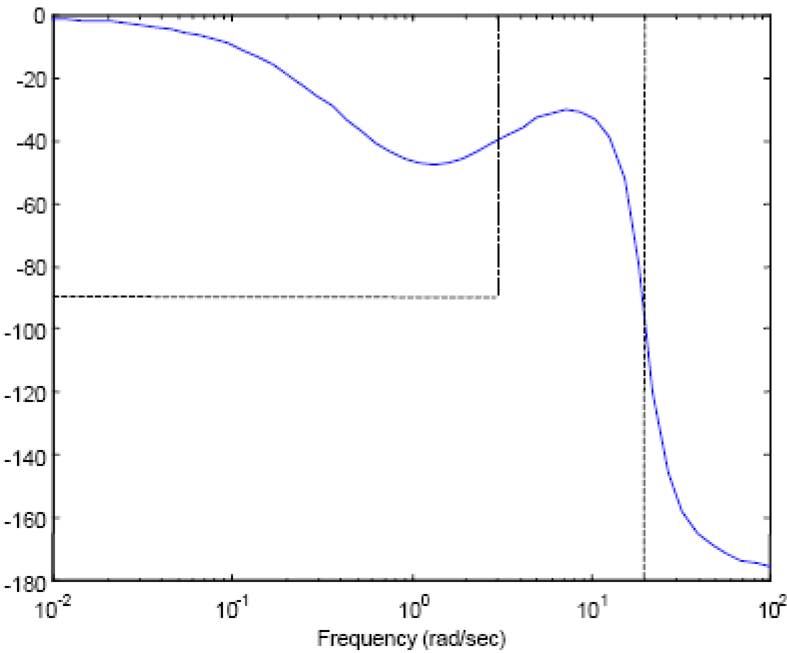
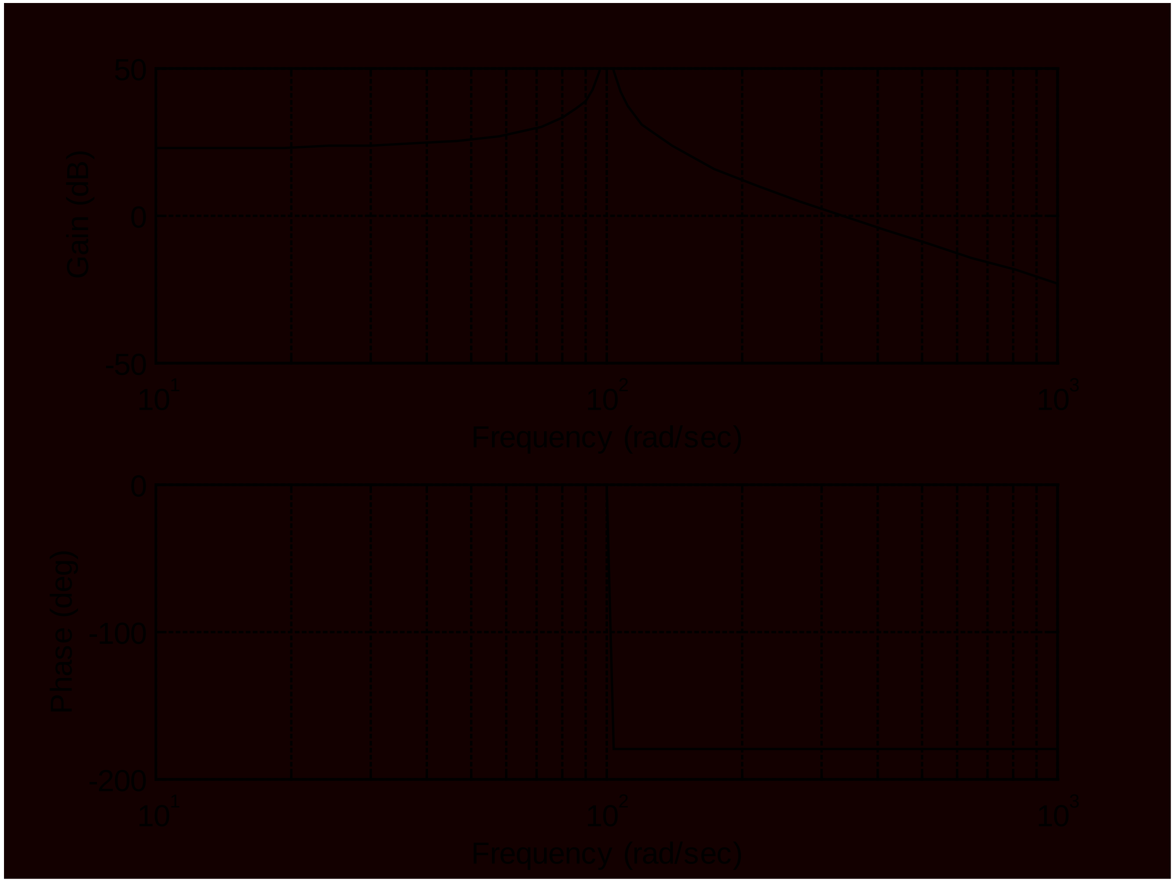


Figure 2. Phase plot

Example: Given below are the frequency response curves for some unknown system.



a) What is the order of the system?
b) Find all standard parameters.
c) If $q_i = 5\sin 1000t$, what is $q_o(t)$ at s.s.?
d) For this system if $q_i(t)$ = step input of 10 units sketch $q_o(t)$. Label axes and show values.

Solution

a) 2nd order $\Rightarrow 180^\circ$ phase shift & -40 dB/decade

b) $20\log k = 20\,\text{dB}$

$\Rightarrow k = 10$

$\omega_n = 100$ rad/s

$\zeta = 0.0$

c) $q_i(t) = 5\sin(1000t)$

$20\log(q_o/q_i) = -20$ ($\omega = 1000$ rad/sec, from figure)

$|q_o/q_i| = 0.1\,\text{rad/sec}$

$\Rightarrow |q_o(i\omega)| = 0.5$

$\text{phase}\,(q_o/q_i) = -180^\circ$

$$\Rightarrow q_o(t) = 0.5\sin(1000t - 180^\circ)$$

d) $q_i(t)$ = step input of 10 units

$$\frac{q_o}{q_i}(s) = \frac{\omega_n^2 k}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Then

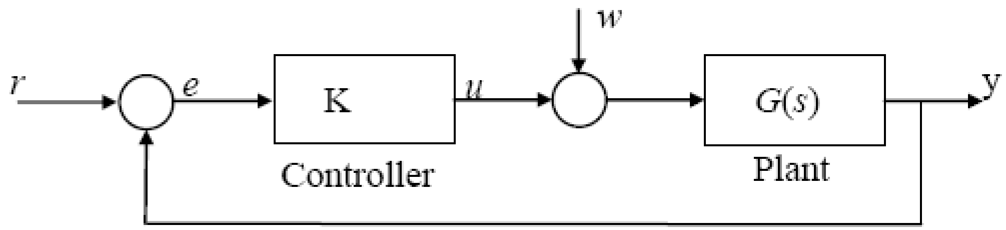
$$\zeta = 0$$

$$k = 10$$

$$\omega_n = 100 \text{ rad/s}$$

$$T = 2\pi / \omega_n = 0.0628 \text{ sec} = \pi / 50 \text{ sec}^{-1}$$

DETERMINING STEADY STATE ERRORS USING A BODE PLOT



How do we calculate ss error using the Bode plot?

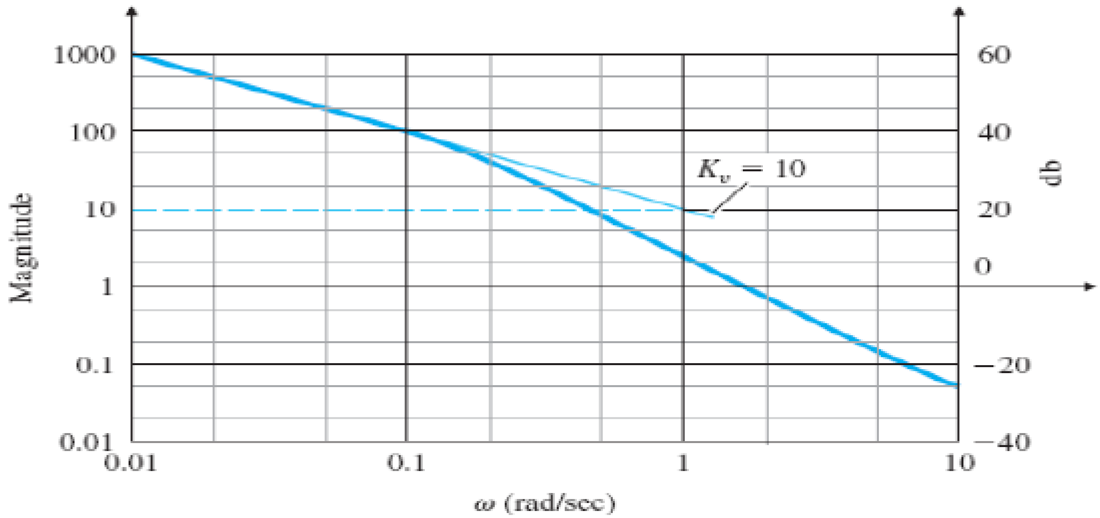
- **Recall from chapter 4 that the ss error for a system decreases as the OL gain increases**
- **Also, at very low frequencies, $KG(j\omega) = K_0(j\omega)^n$ (recall K_0 is the gain when the TF is cast in 'standard' form, and n is the number of free s terms in the denominator of the OLTF).** **Conclusion – the larger the value of the magnitude of the low frequency asymptote, the lower the steady state errors will be for the closed loop system.**
- **When n=0 (type 0 system), the low frequency asymptote is a constant and the gain Ko of the OL is equal to the position error constant K_p. Recall that** $e_{ss} = \frac{1}{1+K_p}$ **for a step input.**

When n=-1 (type 1 system), the low frequency slope has a value of -20 dB/decade, and this can be used to determine K_v directly from the plot. How? $K_v = \lim_{s\to 0} sD(s)G(s)$ **= K_0. But how do we determine K_0 from the Bode plot? …… Remember that the low frequency gain is K_0/ω, and so on the magnitude plot, extend the low frequency asymptote to ω=1 rad/sec and read the value of the magnitude and that will be K_0. Alternately, one could read the magnitude at any frequency on the low-frequency asymptote and compute K_0= K_v=$\omega A(\omega)$, where $A(\omega)$ is the magnitude of the low-frequency asymptote at the frequency ω.**

Computation of K_v

Figure 6.13
Determination of K_v from the Bode plot for the system $KG(s) = 10/[s(s+1)]$

As an example of the determination of steady-state errors, a Bode magnitude plot of an open-loop system is shown in Fig. 6.13. Assuming that there is unity feedback as in Fig. 6.4, find the velocity-error constant, K_v.



Solution. Because the slope at the low frequencies is −1, we know that the system is type 1. The extension of the low-frequency asymptote crosses $\omega = 1$ rad/sec at a magnitude of 10. Therefore, $K_v = 10$ and the steady-state error to a unit ramp for a unity feedback system would be 0.1. Alternatively, at $\omega = 0.01$ we have $|A(\omega)| = 1000$; therefore, from Eq. (6.23) we have

$$K_o = K_v \cong \omega|A(\omega)| = 0.01(1000) = 10.$$

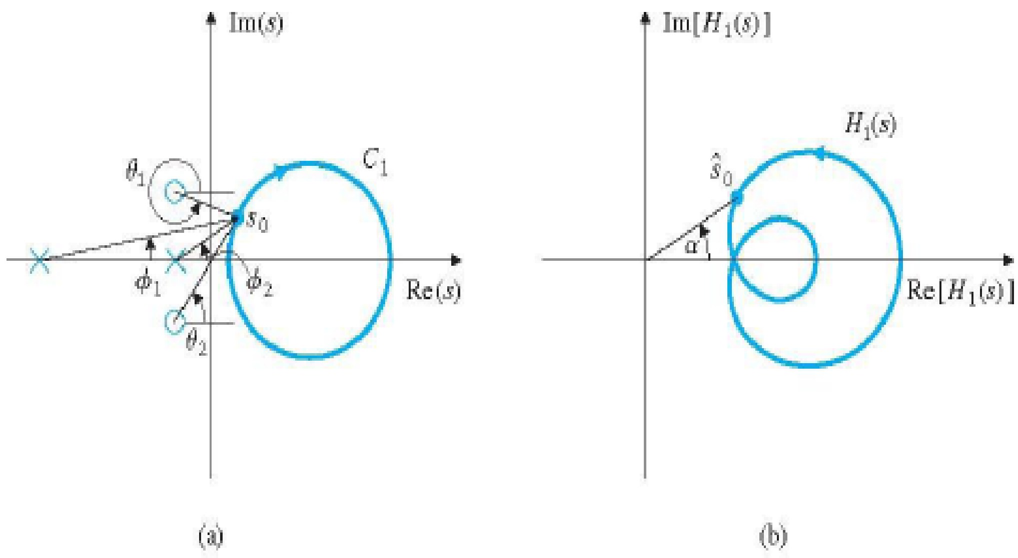
6.3. CONTOUR MAPS

What is meant by a contour map of G(s) = 1/(s+1)?
Answer: Vary s along the contour (whatever it may be) and plot how G(s) varies. This is the contour map of G(s). (also see Appendix B of text)

An important observation about contour maps was found long long back.....termed the Argument principle.

Argument Principle: A contour map of a complex function will encircle the origin Z-P times, where Z is the number of zeros and P is the number of poles of the function inside the contour. If the contour 'direction' is clockwise, the encirclement will be clockwise for a zero, and counter-clockwise for a pole.

s-contour (left) and contour map (right) for some G(s) given in pole-zero form



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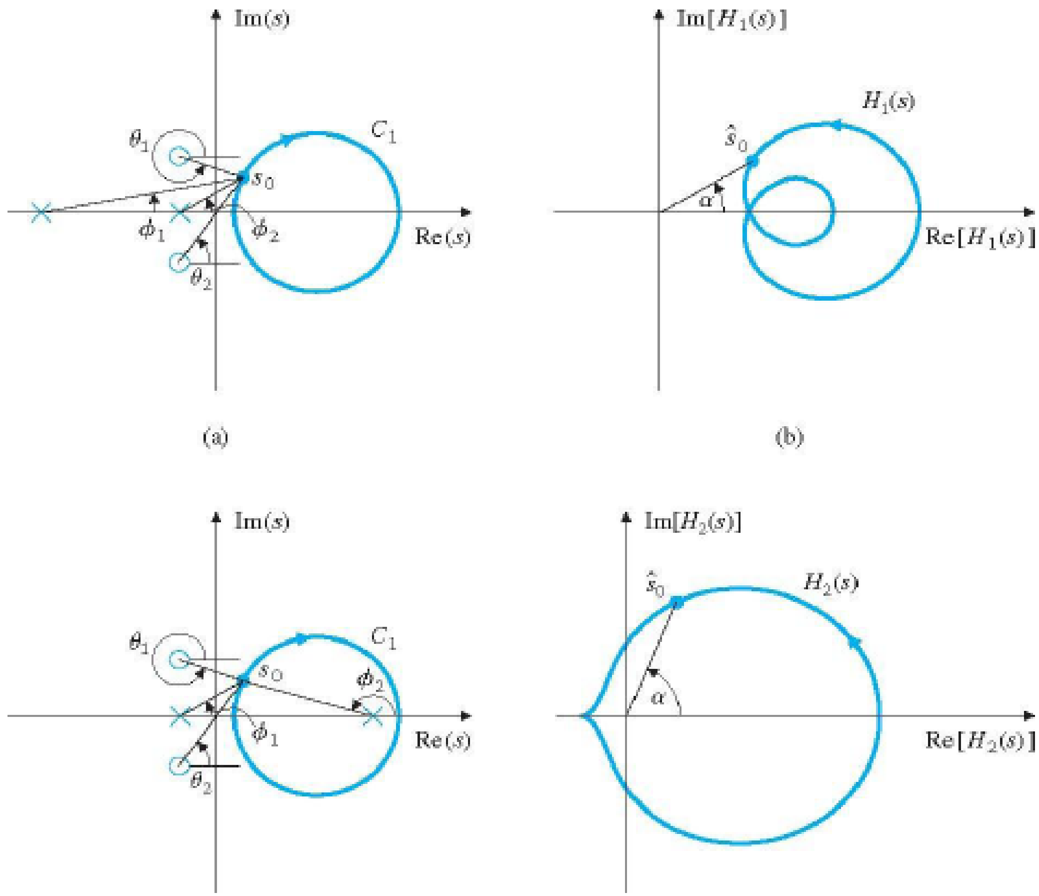
s-contour (left) and contour map (right) for some G(s) given in pole-zero form



Note that the argument of G(s) is $\alpha = \theta_1 + \theta_2 - (\phi_1 + \phi_2)$. As s traverses C_1 in the clockwise direction starting at s_0, the angle α of G(s) will change (decrease or increase), but it will not undergo a net change of 360^0 as long as there are no poles or zeros within contour C_1. This is because none of the angles that make up α go through a net revolution. This means that the contour plot of G(s) will not encircle the origin – the key result that Nyquist used for control design.

Draw a contour map for the function G(s) = (s-1)/[(s+3)(s+6)] , using a unit circle centred around (1,0) as the s-contour?

How do we move to control design from this simple contour map example? To do this, we first need to the 'NYQUIST PLOT'. Consider the system $G(s) = K/(s+1)$. **For any open loop TF KG(s), if the entire right half plane is selected as the s-contour, the resulting map of KG(s) is called the Nyquist plot, i.e., it is a contour map of the OLTF for that specific contour.**

Draw the Nyquist plot for OLTF $G(s) = K/(s+1)$ and for $G(s) = K\,(s+2)/(s+10)$:

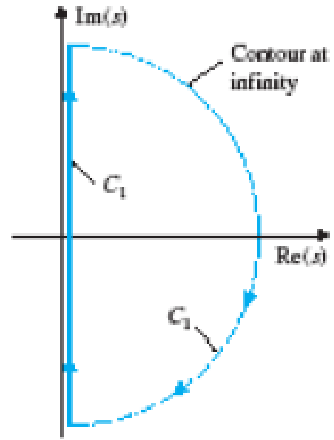
How? …..Plot the OLTF KG(s) for $-j\infty \leq s \leq +j\infty$. Do this by first evaluating KG(jω) for ω=0 to ω_h, where ω_h is so large that the magnitude of KG(jω) is negligibly small for $\omega > \omega_h$, then reflecting the image about the real axis and adding it to the preceding image. The magnitude of KG(jω) will be small at high frequencies for any physical system. The Nyquist plot will always be symmetric with respect to the real axis.

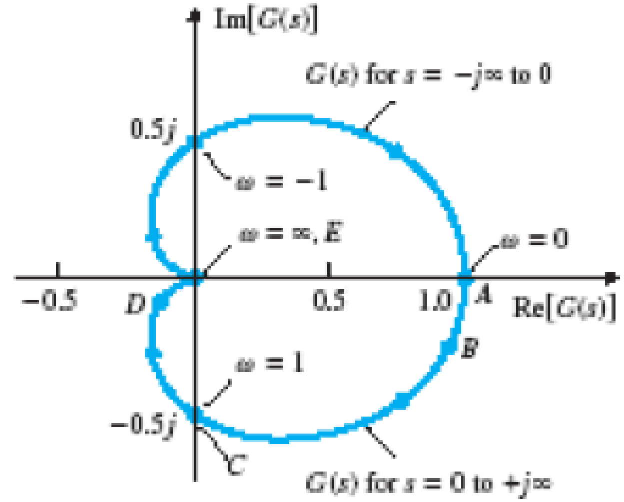
Next, draw the Nyquist plot for the OLTF $G(s) = K/[s(s+3)(s+5)]$:

6.4. NYQUIST CRITERION

Fundamentals

Argument Principle: A contour map of a complex function will encircle the origin Z-P times, where Z is the number of zeros and P is the number of poles of the function inside the contour. If the contour 'direction' is clockwise, the encirclement will be clockwise for a zero, and counter-clockwise for a pole.





How can we use Argument Principle to help us determine stability???
Considering the CLTF, we know that the CL poles are the roots of the CE: 1+ KG(s) = 0. **Our objective is to determine whether 1 + KG(s) has any right half plane zeros.**

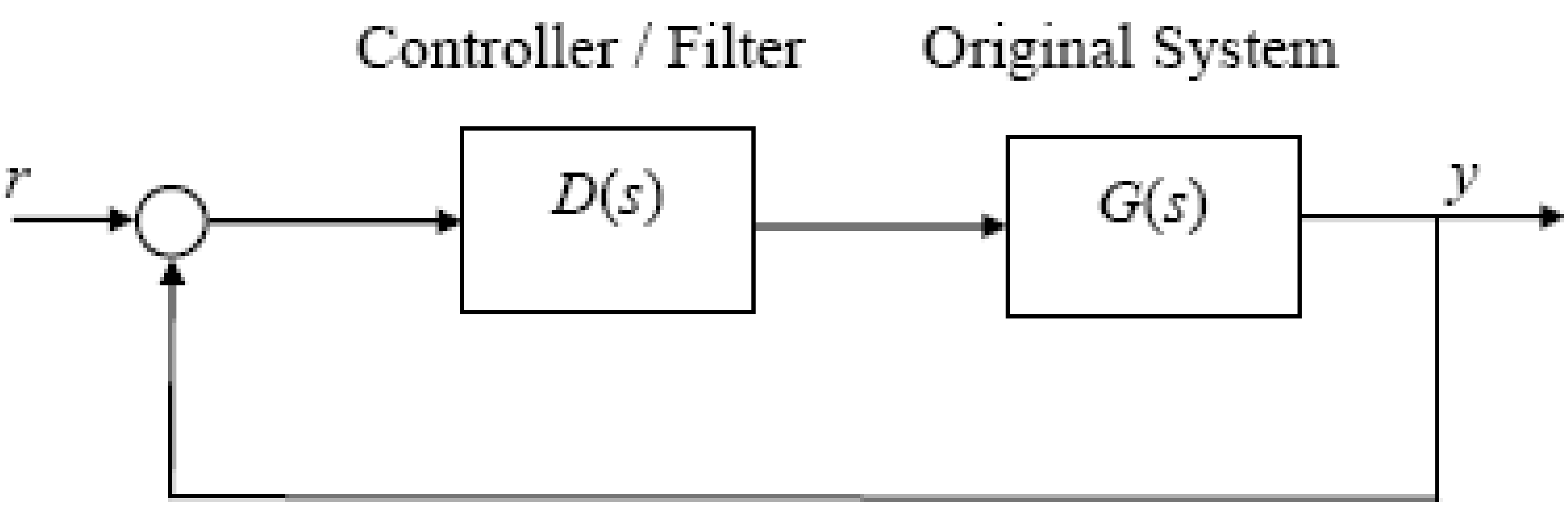
Connection: Consider a 'clockwise' contour C that encircles the entire right half plane. Select the function H(s) = 1 + KG(s) and, using the principle cited, check if H(s) encircles the origin to determine how many poles and zeros of H(s) are in the right half plane! Applied to this case, the principle says "The contour map of a 1+KG(s) will encircle the origin Z-P times clockwise, where Z is the number of zeros and P is the number of poles of 1+KG(s) in the right half plane".

Since we typically know how many poles H(s) has in the right half plane (see below), this will tell us how many zeros H(s) has in the right half plane.
BUT, before that, let us do one more trick.
Checking if H(s) = 1+ KG(s) encircles the origin is the same as checking if KG(s) encircles the point (-1,0) !
It is easier for us to work with the OL transfer function KG(s) rather then 1+ KG(s), so we will be checking whether KG(s) encircles the point (-1,0). So, plot KG(s) as 's' varies along the contour C , i.e., get the Nyquist plot.

What is Nyquist Criterion?

The Nyquist Criterion relates the stability of a closed-loop system to the open-loop frequency response and open-loop pole locations. Thus, knowledge of the open-loop system's frequency response yields information about the stability of the closed-loop system. **The Nyquist criterion can be stated as follows:**

For an open loop system KG(s), the number of zeros Z of 1+KG(s)=0 in the RHP (i.e., no. of closed loop poles P in the RHP) = Number of clockwise encirclements N of (-1,0) by the s-contour that covers the entire RHP + No. of poles of KG(s) (open loop poles). In other words, Z = N+P.



Derivation of the Nyquist Criterion

The Nyquist criterion can tell us how many closed-loop poles are in the right-half plane. Four important concepts that will be used during the derivation (note: KG(s) below can be replaced with KG(s)H(s) in case there is a transfer function H(s) in the feedback look, i.e. the OL transfer function is what is needed) are as follows, some of which were explained above:

(1) the relationship between the poles of $1+KG(s)$ and the poles of $KG(s)$;
(2) the relationship between the zeros of $1+KG(s)$ and the poles of the closed-loop transfer function, $T(s)$;
(3) the concept of mapping points;
(4) the concept of mapping contours.

Consider the following notation

$$G(s) = \frac{N}{D}$$

$$1 + KG(s) = 1 + \frac{KN}{D} = \frac{D + KN}{D}$$

$$T(s) = \frac{KG(s)}{1 + KG(s)} = \frac{KN}{D + KN}$$

We can conclude from these expressions that:

(1) the poles of $1+KG(s)$ are the same as the poles of $G(s)$, the open-loop system (which we know typically!);
(2) the zeros of $1+KG(s)$ are the same as the poles of $T(s)$, the closed-loop system (which are to be determined).

PROCEDURE FOR DETERMINING NYQUIST STABILITY

1. Plot the OLTF KG(s) for $-j\infty \le s \le +j\infty$. Do this by first evaluating $KG(j\omega)$ for $\omega=0$ to ω_h, where ω_h is so large that the magnitude of $KG(j\omega)$ is negligibly small for $\omega > \omega_h$, then reflecting the image about the real axis and adding it to the preceding image. The magnitude of $KG(j\omega)$ will be small at high frequencies for any physical system. The Nyquist plot will always be symmetric with respect to the real axis.

2. Evaluate the number of encirclements of (-1,0), and call that number N. Do this by drawing a straight line in any direction from (-1,0) to ∞. Then count the net number of left-to-right crossings of the straight line by KG(s). If encirclements are in the counterclockwise direction, N is negative.

3. Determine the number of unstable (RHP) poles of G(s), i.e., open loop poles, and call that number P.

4. Calculate the number of unstable closed-loop roots Z as $Z = N+P$, where P is the number of open loop poles of G(s).

Obviously, for stability, we need $Z = 0$, i.e., no RHP poles of the closed loop TF. Note that this procedure allows you to determine the stability of the closed loop, using the open loop transfer function G(s).

What K will keep the system stable for the following OL systems?

$G(s) = K/(s+1)$ and for $G(s) = K\,(s+2)/(s+10)$

Problem: For the unity feedback system where $G(s) = K/[s(s+3)(s+5)]$, find the gain K, for stability, instability, and the value of gain for marginal stability. For marginal stability, also find the frequency of oscillation. Use the Nyquist criterion.

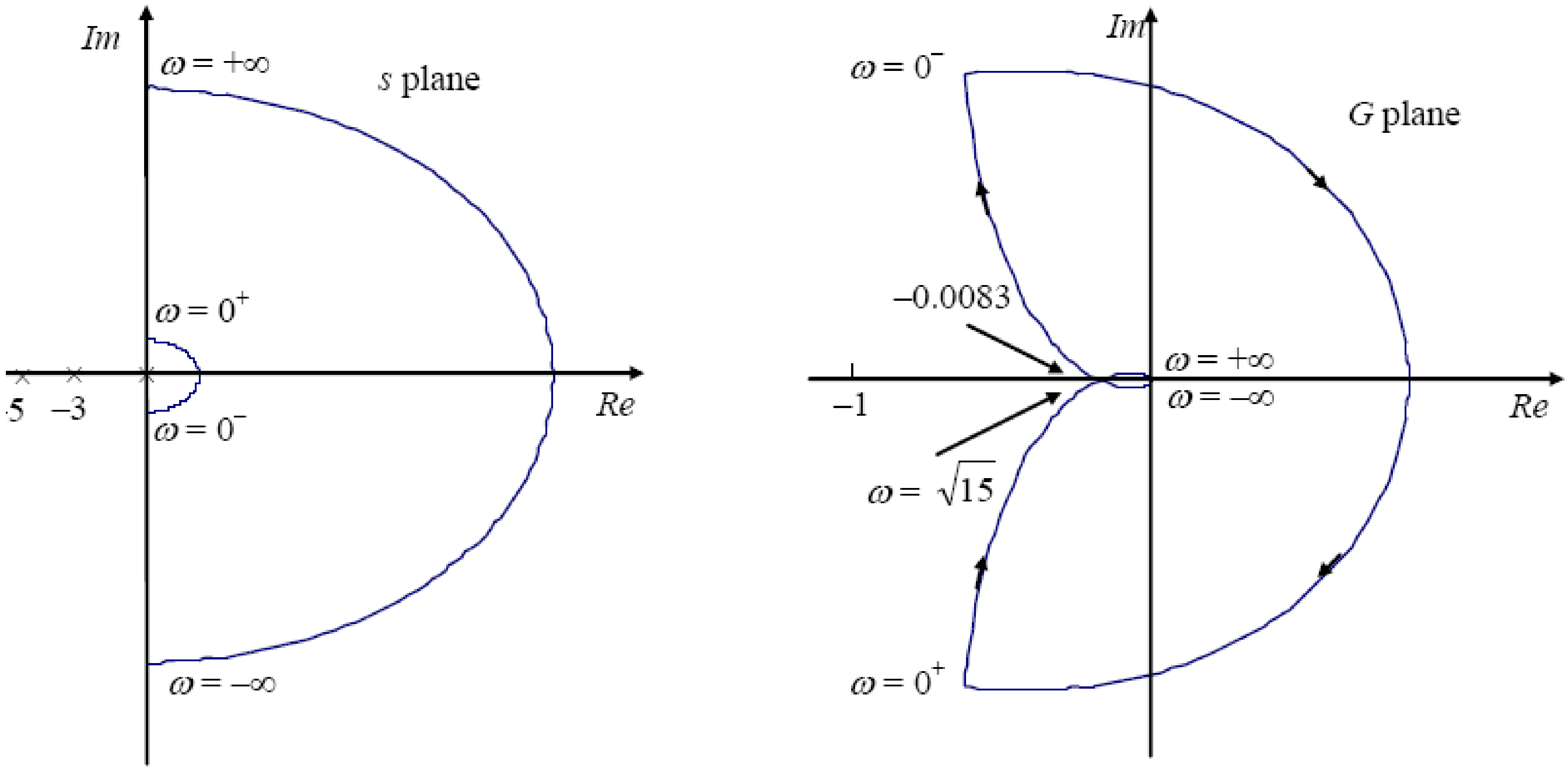
Solution: First, set $K = 1$ and sketch the Nyquist diagram for the system, using the contour. For all points on the imaginary axis,

$$G(j\omega) = \left.\frac{K}{s(s+3)(s+5)}\right|_{K=1, s=j\omega} = \frac{-8\omega^2 - j(15\omega - \omega^3)}{64\omega^4 + \omega^2(15-\omega^2)^2} \qquad \text{(e.1)}$$

Also, at $\omega = 0$, $G(j\omega) = -0.0356 - j\infty$.

Next, find the point where the Nyquist diagram intersects the negative real axis. Setting the imaginary part of Eq. (e.1) equal to zero, we find $\omega = \sqrt{15}$ rad/s = 3.87 rad/s. Substituting this value of ω back into Eq. (e.1) yields the real part as –0.0083. Finally, at $\omega = \infty$, $G(j\omega)H(j\omega) = 1/(j\infty)^3 = 0\angle{-270^\circ}$. For stability, N must then be equal to zero.

The system is stable if the critical point (-1,0) lies outside the contour ($N = 0$), so that $Z = P - N = 0$. Thus, K can be increased by 1/0.0083 = 120.48 before the Nyquist diagram encircles (-1,0). Hence, for stability, K<120.48. For marginal stability, $K = 120.48$. At this gain, the Nyquist diagram goes through the critical poir (–1,0), and the frequency of oscillation is $\omega = 3.87$ rad/sec.



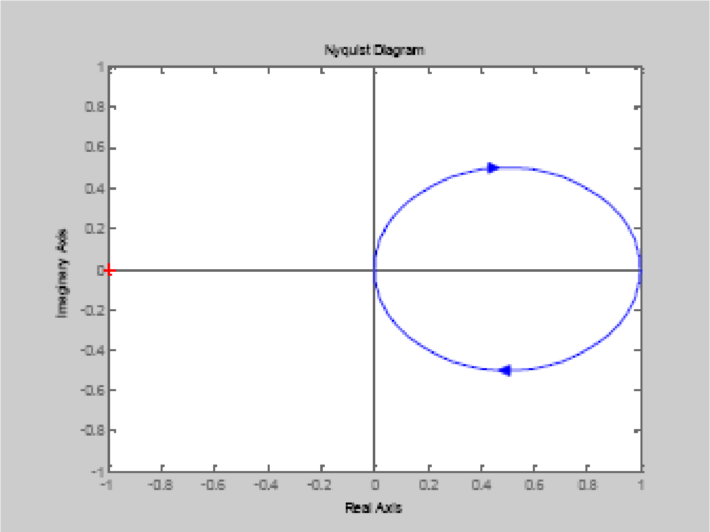
EXAMPLES OF NYQUIST PLOTS

(note: you plot the OLTF and infer the properties of the closed loop TF!)

The relation between the gain and phase has been thought of as the same for all the systems in the early days i.e., an increasing gain causes instability but it was found later that it may not be the case every time, and the relationship may be reversed. Nyquist studied the occasional reversals and came up with a key finding which is now known as the Nyquist stability criterion. The Nyquist stability criterion relates the open loop frequency response to the number of closed loop poles of the system in the RHP. Study of the Nyquist criterion will allow you to determine stability from the frequency response of a complex system, perhaps with one or more resonances, where the magnitude curve crosses 1 several times and/or the phase crosses 180^0 several times. It is also very useful in dealing with the open-loop, unstable systems, nonminimum-phase systems, and systems with pure delays.

Examples:

1) OLTF = $\dfrac{1}{(s+1)}$

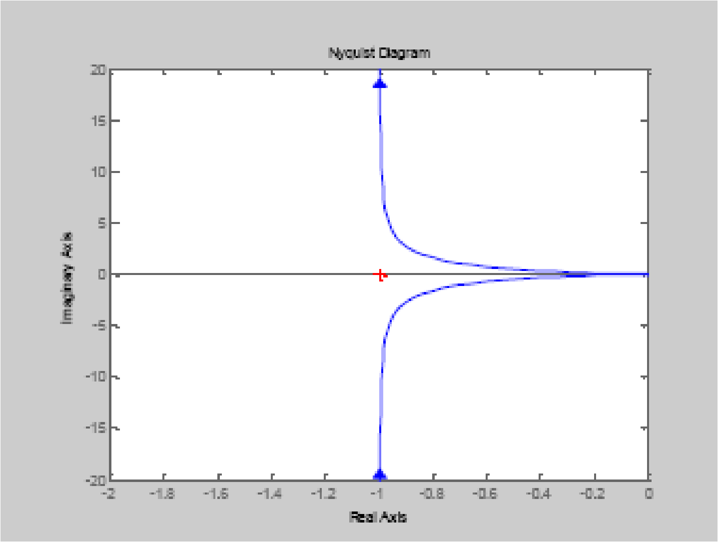


CODE:

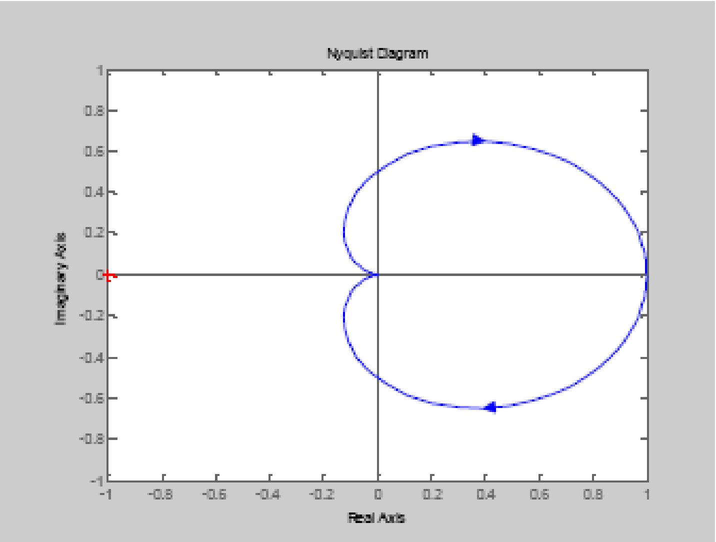
```
num = [1];
den = [1 1];
sys = tf(num,den);
nyquist(sys);
```

[NOTE: the critical point (-1,0)is shown in RED in the plots below]

2) OLTF= $\dfrac{1}{s(s+1)}$ The code for this is similar to the above code except that you have to change the numerator and the denominator values.



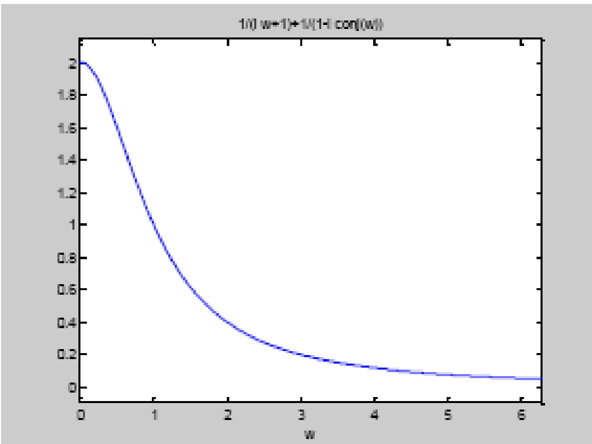
3)OLTF= $\dfrac{1}{(s+1)^2}$



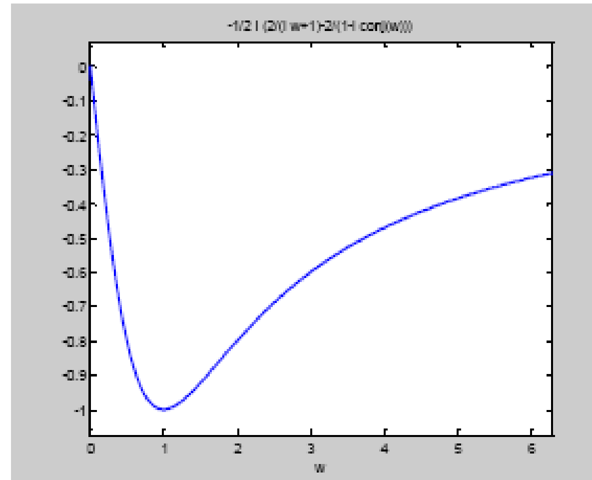
Looking at a TF with various lenses

1) $\dfrac{2}{(s+1)}$

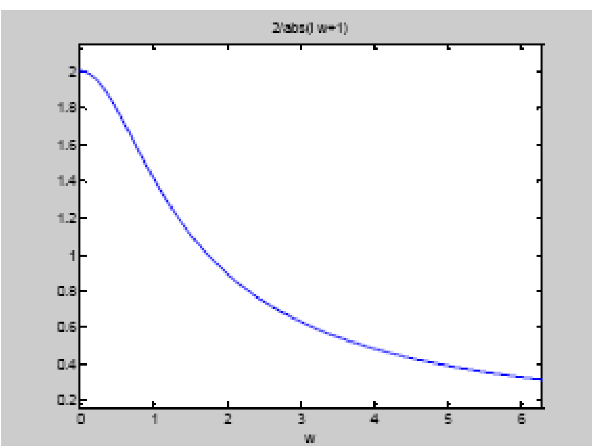
Real part Vs w



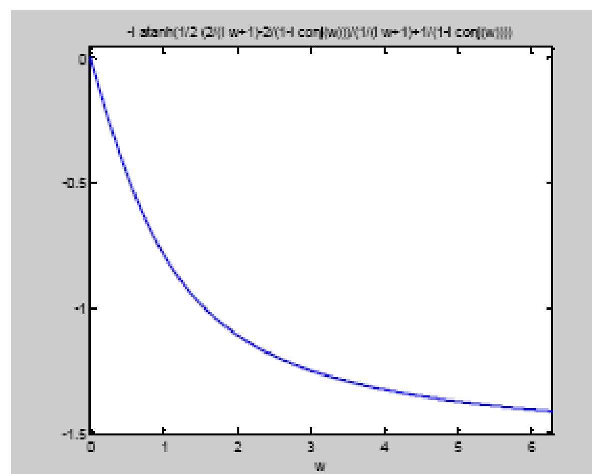
Imaginary part Vs w



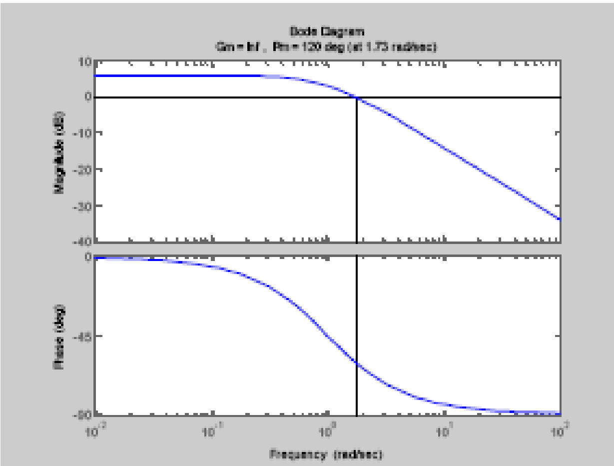
Magnitude Vs w



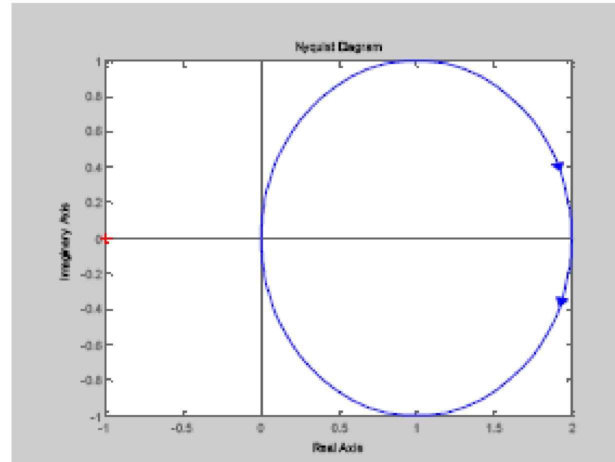
Phase Vs w



Bode plot



Nyquist plot



The code for all the following examples will follow the code given below by just changing the function in the code.

CODE:

```
syms w
sys =2/(j*w+1);
x = real(sys);
figure(1);
ezplot(x)
y = imag(sys);
figure(2);
ezplot(y)
z=abs(sys);
figure(3);
ezplot(z);
figure(4);
k = atan(y/x);
ezplot(k);
```

http://www.facstaff.bucknell.edu/mastascu/eControlHTML/Freq/Nyquist3.html#BodeI
nterpretation

http://www.facstaff.bucknell.edu/mastascu/eControlHTML/Freq/Nyquist4.html

6.6. STABILITY MARGINS

Most control systems are stable for small gain values and become unstable with increase in gain past a certain critical value. The two commonly used quantities that measure the stability margin for such systems are the gain margin and the phase margin both of which are directly related to the stability criterion.

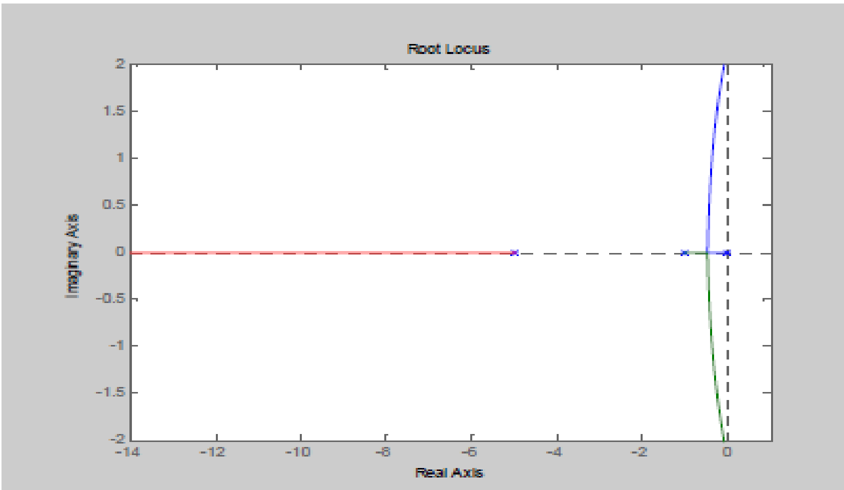
The **gain margin (GM)** is the factor by which gain can be raised before instability results. For the typical case, it can be read directly from the Bode plot by measuring the vertical distance between the $|KG(j\omega)|$ curve and the $|KG(j\omega)| = 1$ (i.e., 0 dB) line at the frequency where phase of $G(j\omega) = 180^0$. This value will be in dBs which can be converted to a 'factor' using "vertical distance (dBs) = 20 log (factor)". So, GM is the factor by which the gain K can be raised before instability results. Therefore, $|GM| < 1$ (or $|GM| < 0$ dB) indicates an unstable system, i.e., the $|KG(j\omega)|$ curve cannot be above the 0 dB line at the frequency where angle of of $G(j\omega) = 180^0$. The GM can also be determined from a root locus with respect to K by noting two values of K: (1) at the point where the locus crosses the $j\omega$-axis, and (2) at the nominal closed-loop poles. The GM is the ratio of these two values

Another measure that is used to indicate the stability margin in a system is the **phase margin (PM)**. It is the amount by which the phase of $G(j\omega)$ exceeds -180^0 when $|KG(j\omega)| = 1$, which is an alternative way of measuring the degree to which the stability conditions are met. A positive PM is required for stability. On the Bode plot, find the phase where the gain curve $|KG(j\omega)|$ curve crosses the 0 dB line. If that is –x degrees, then PM = (180-x) degrees, i.e., the system can accommodate that much 'pure phase' before instability, i.e., pushing the phase curve below the -180 degree line.

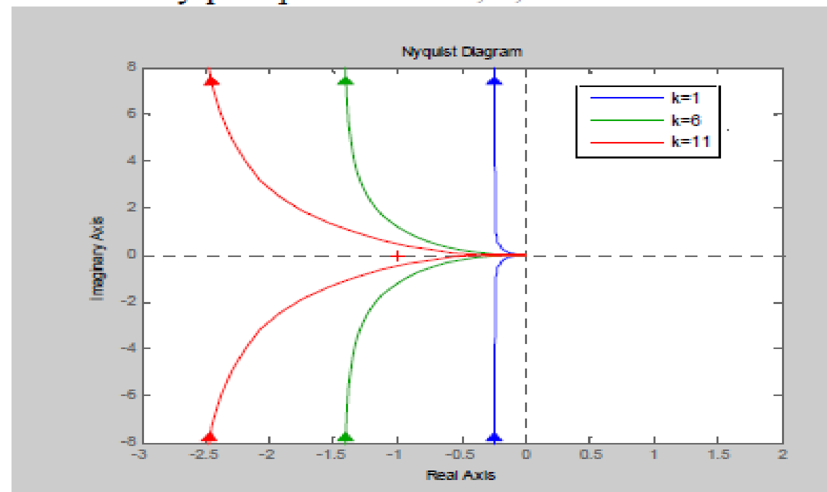
Sketch the root locus, Nyquist and Bode plots for (i) a type 0, third order system, and (ii) type 1 third order system. Then identify gain and phase margins.

$$kG(s) = \frac{k}{s(s+1)(s+5)}$$

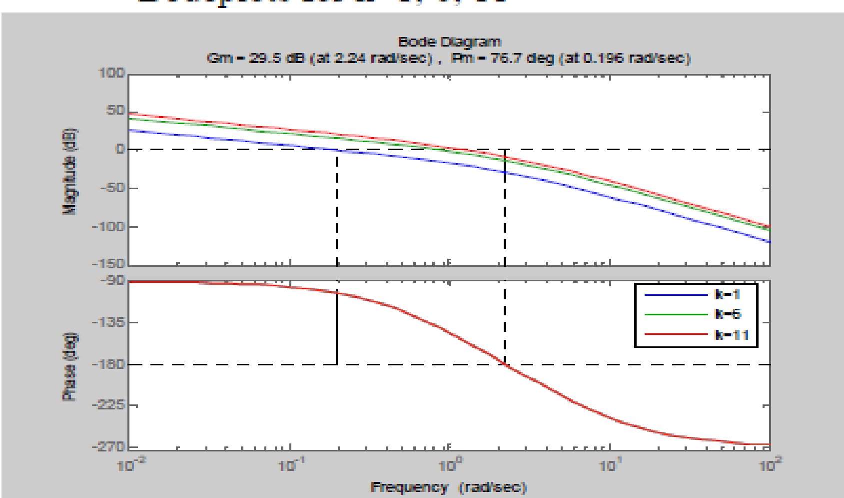
Root locus



Nyquist plots for k=1, 6, 11

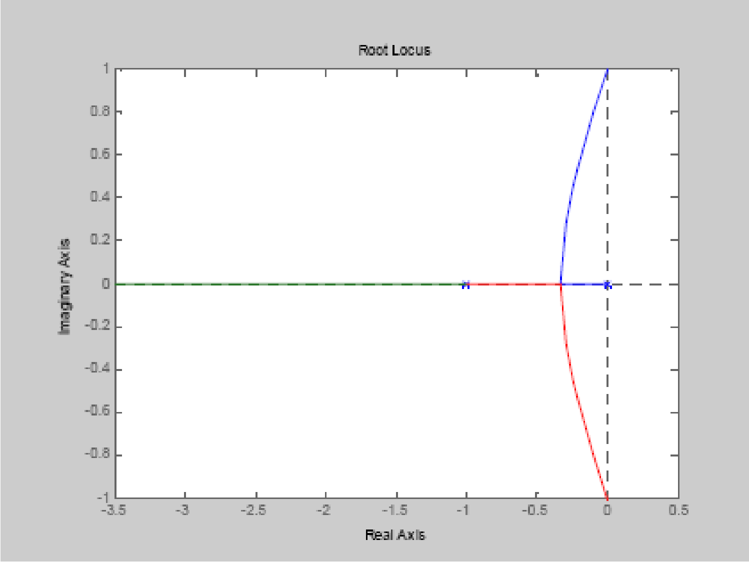


Bodeplots for k=1, 6, 11

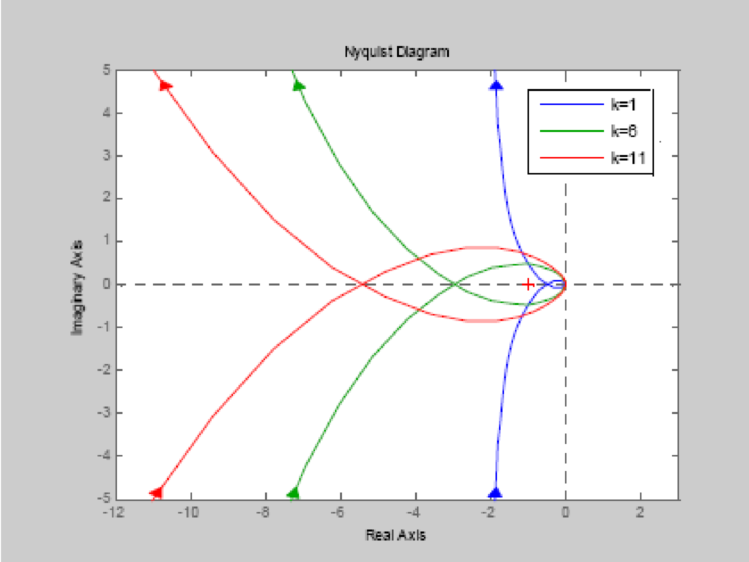


$$kG(s) = \frac{k}{s(s+1)^2}$$

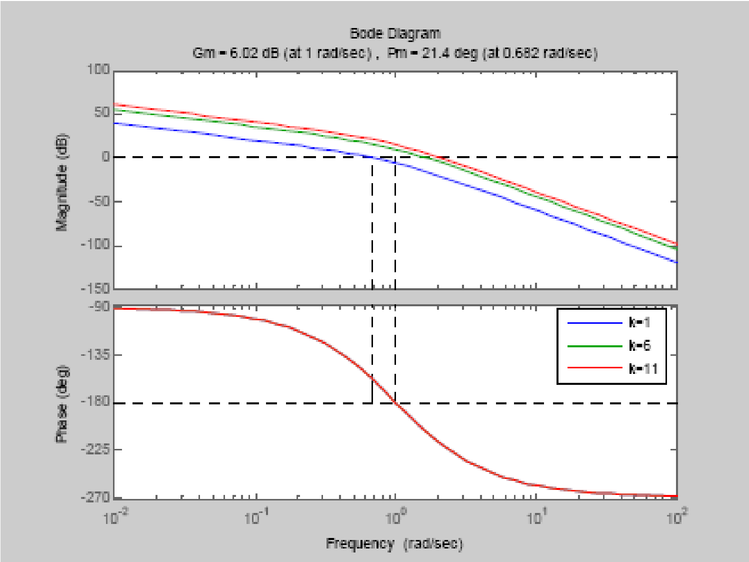
Root locus



Nyquist plots for k=1, 6, 11



Bodeplots for k=1, 6, 11



The stability margins of different systems are shown as below: The (-1,0) point plays the same role in the Nyquist diagram as the imaginary axis does in the in the root locus diagram. For a stable system, the closer the Nyquist curve approaches the (-1,0) point, the less stable the system is, as shown in the figure below.

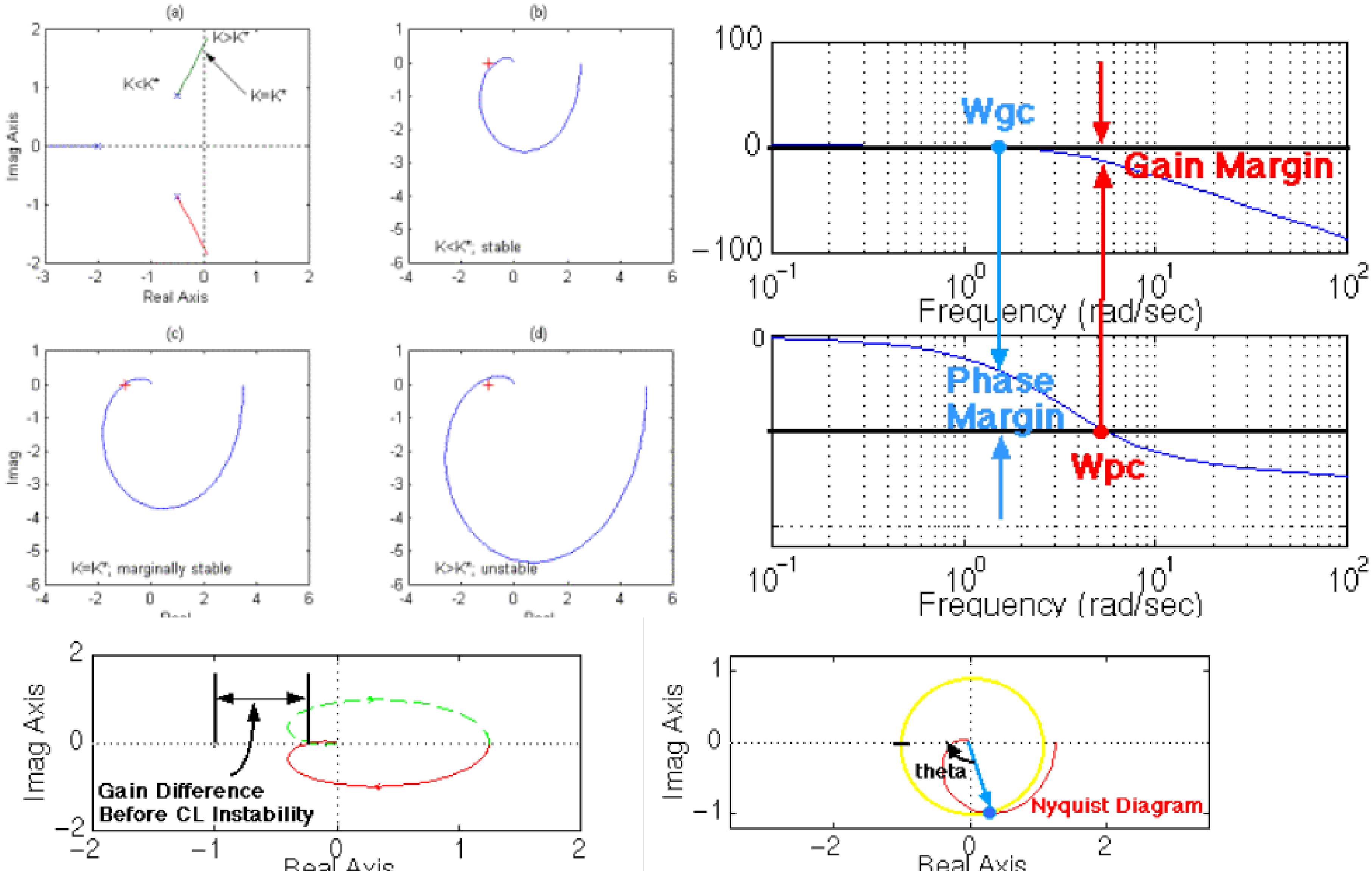


Figure 1. (a) Relationship between imaginary axis in root locus diagram and (-1,0) point in Nyquist diagram; (b) Gain and phase margins in a Bode plot; (c) GM and PM on a Nyquist plot

RECALL the time domain specs

Formulae for some of transient specifications for the *underdamped* case are as follows:

Time Domain:

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}; \qquad M_p = e^{-\zeta\pi / \sqrt{1-\zeta^2}} \times 100\% \quad (\%OS = M_p * 100); \qquad \sigma = \zeta\omega_n$$

$$T_s = \frac{4.6}{\zeta\omega_n} \text{ (for 1\% settling band)}; \quad T_r = \frac{1.8}{\omega_n} \text{ (for } \zeta = 0.5\text{)}; \qquad \omega_d = \omega_n \sqrt{1-\zeta^2}$$

$\omega_n \equiv$ undamped natural frequency; $\zeta \equiv$ damping ratio; $T_p \equiv$ peak time; $T_s \equiv$ settling time; $T_r \equiv$ rise time.

How these related to the gain and phase margins?....and to the bandwidth?

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}; \quad \omega_p = \omega_n\sqrt{1-2\zeta^2}; \quad \omega_{BW} = \frac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{2+4\zeta-4\zeta^2}}$$

$$\Phi_M = 90^\circ - \tan^{-1}\frac{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}{2\zeta} = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}; \qquad \zeta \approx \frac{\Phi_M}{100}$$

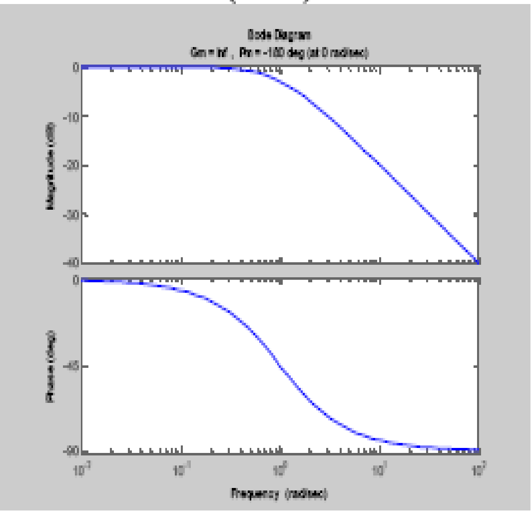
$$\omega_{BW} = \omega_c \text{ for } PM = 90^\circ; \qquad \omega_{BW} = 2\omega_c \text{ for } PM = 45^\circ$$

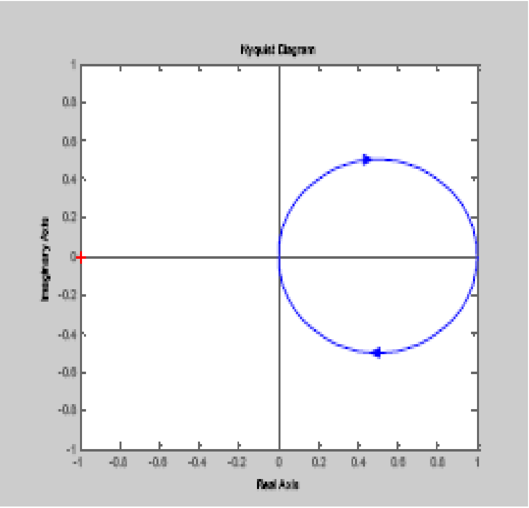
$M_p \equiv$ peak overshoot at frequency ω_p; $\omega_{BW} \equiv$ system bandwidth (0.707of max value, or –3dB down from max value at $\omega = 0$); $\Phi_M \equiv$ phase margin, PM; $\omega_c \equiv$ crossover frequency (at the magnitude = 1 point).

WE WILL USE THESE FOR CONTROLLER (COMPENSATOR) DESIGN IN THIS CHAPTER

SYSTEM | **BODE PLOT (K=1)** | **NYQUIST PLOT**

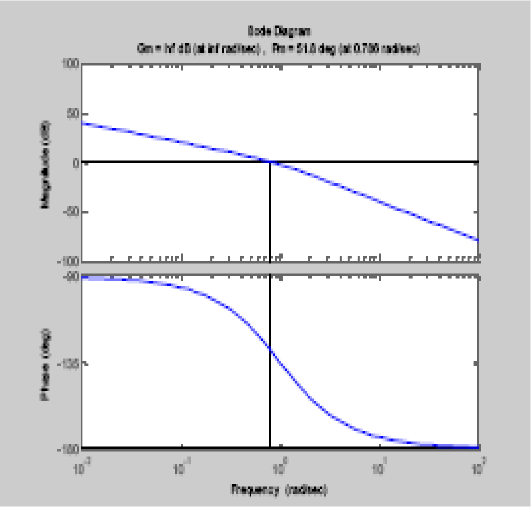
1) $\frac{1}{(s+1)}$

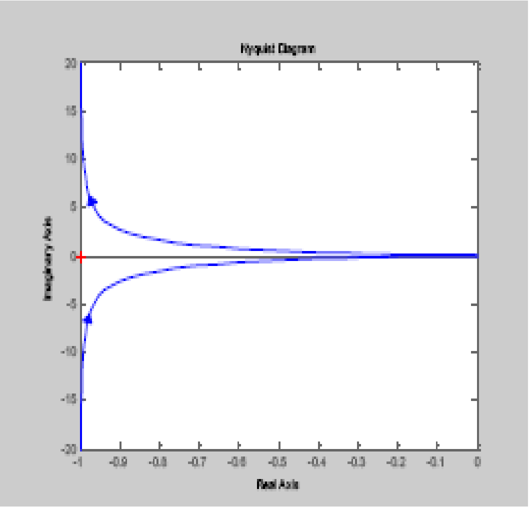




GM = Infinity PM=-180 deg

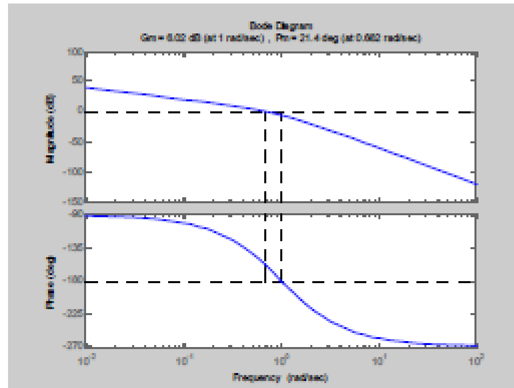
2) $\frac{1}{s(s+1)}$

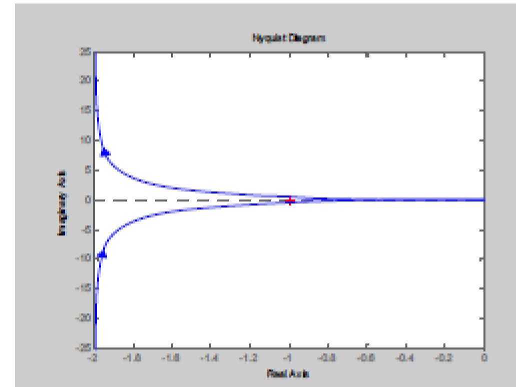




GM = Infinity PM=51.8 deg

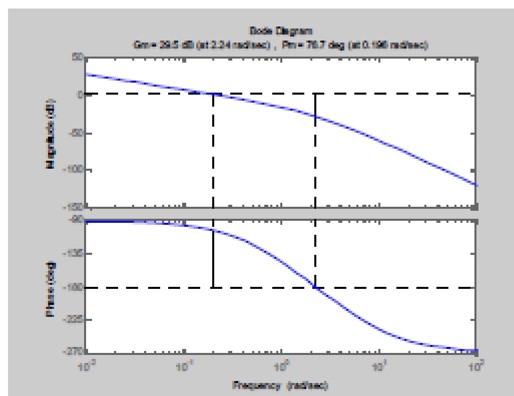
4) $\frac{1}{s(s+1)^2}$

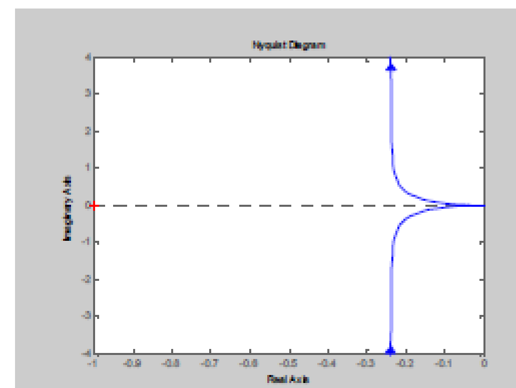




GM = 6.02dB PM=21.4 deg

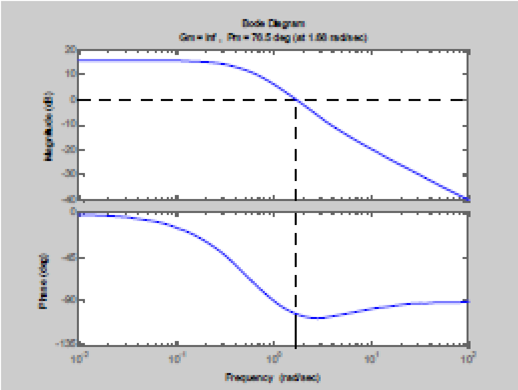
5) $\frac{1}{s(s+1)(s+5)}$

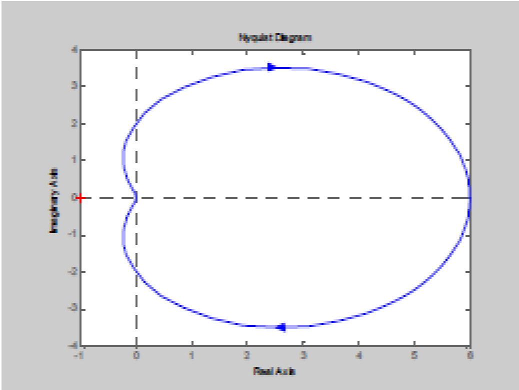




GM = 29.5dB PM=76.7 deg

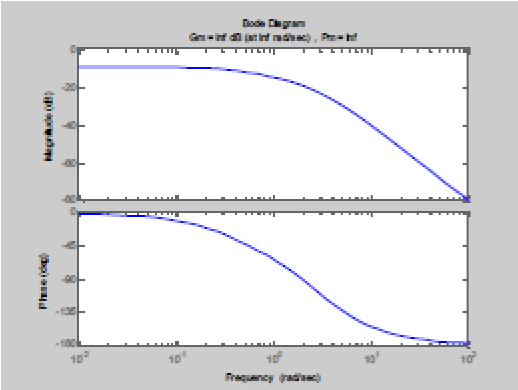
6) $\dfrac{(s+3)}{(s+1)(s+0.5)}$

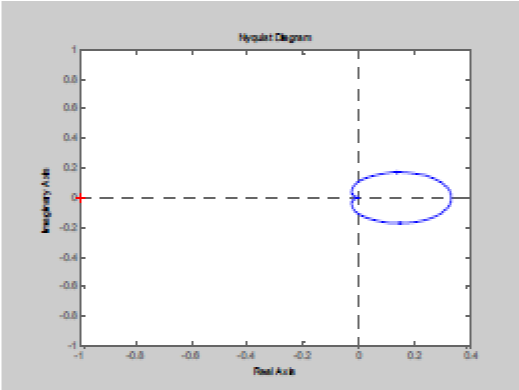




GM = infinity PM=76.5 deg

7) $\dfrac{(s+1)}{(s+0.5)(s+2)(s+3)}$





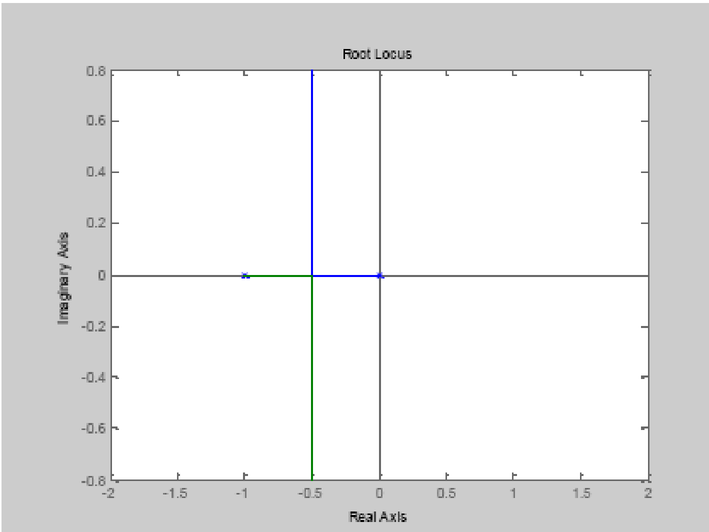
GM = infinity PM=infinity

6.7. COMPARISON BETWEEN ROOT LOCUS, BODE AND NYQUIST PLOTS

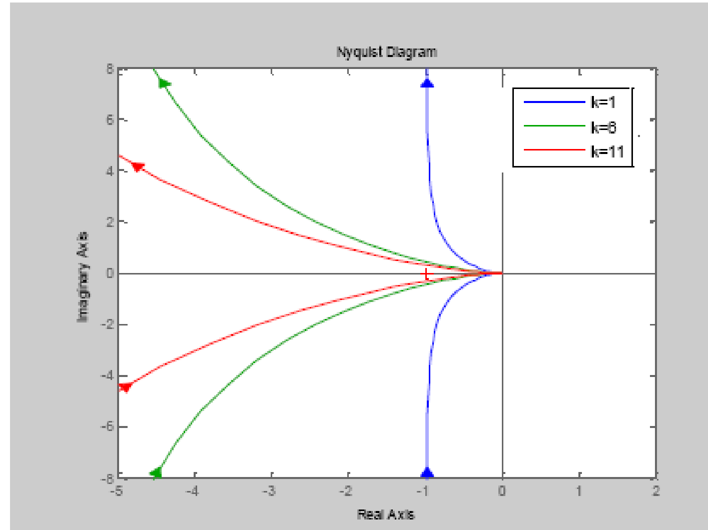
(NOTE: the Nyquist plots below do not include regions at infinity, e.g., the infinite semi-circle in the first plot below)

$$1)\quad kG(s) = \frac{k}{s(s+1)}$$

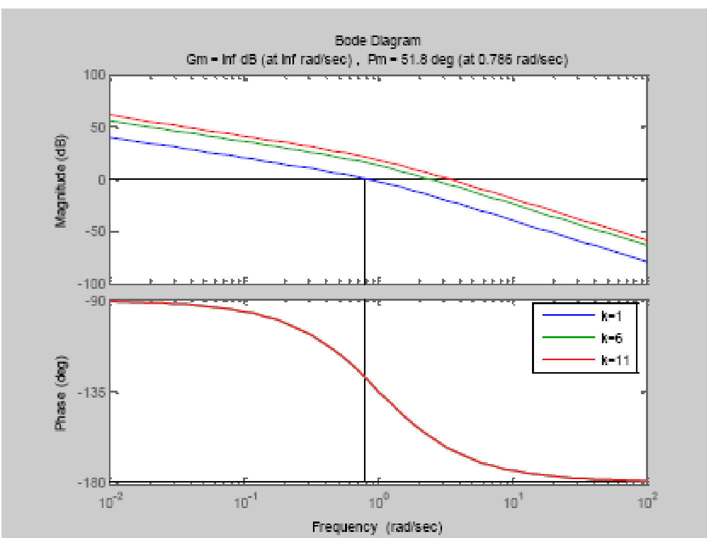
Root locus



Nyquist plots for k=1, 6, 11



Bodeplots for k=1, 6, 11



- The root locus plot shows the locus at different k values.
- The Nyquist plot by default uses a value of k=1 here we show the Nyquist plot for different k values where you can see that even with the variation of the k values it is always consistent with the root locus plot i.e., it never encircles (-1,0).
- When we plot the Bode plots for different k values you can see that the magnitude plot changes but the phase plot remains the same. You can see that the increase in the value of gain i.e., the k value the plot shifts towards the right which shows that the gain crossover shifts to the right but the phase doesn't change much.
- The code shown below gives the gain margin and the phase margin for $k = 1$. You can try to find GM and PM for different values of k just by a slight modification in the code (Change the if statement).

The values of the GM and PM will be displayed on top of the graph for that particular *k* value. For k=1 the GM = infinity and PM = 51.8deg. For *k* = 6 the GM = infinity and PM =23.1deg. For *k*=11 the GM= infinity and PM = 17.1deg

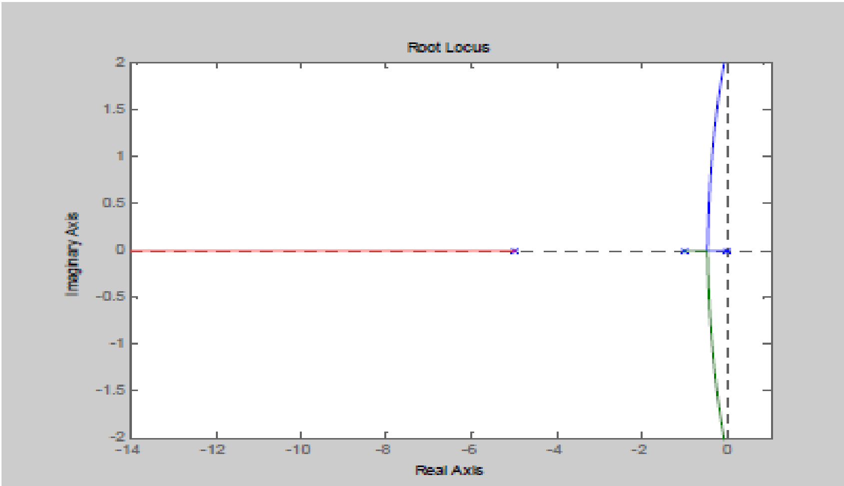
Matlab code:

```
num = 1;
den = [1 1 0];
sys = tf(num,den);
figure(1);
rlocus(sys);
for k = 1:5:11;
num = k;
den = [1 1 0];
sys = tf(num,den);
figure(2);
nyquist(sys);
 hold all;
end
hold off;
for k = 1:5:11;
num = k;
den = [1 1 0];
sys = tf(num,den);
figure(3);
bode(sys);
if k==1
margin(sys);
[Gm,Pm,Wcg,Wcp]=margin(sys);
end
hold all;
end
hold off;
```

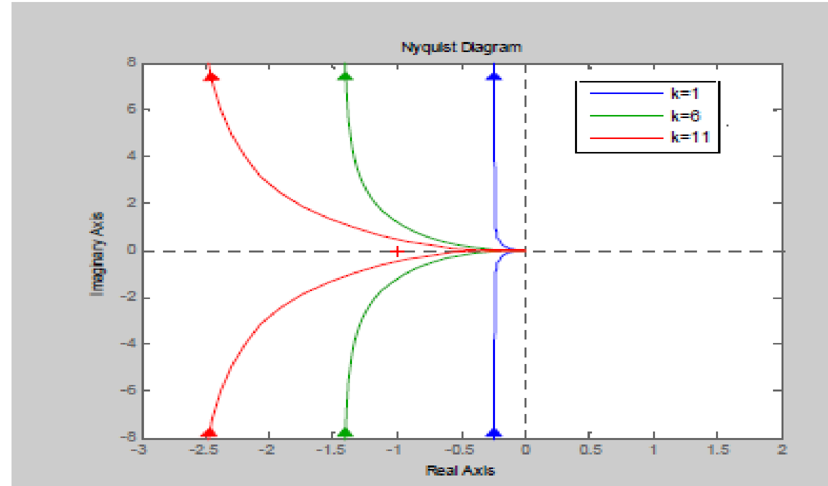
- The above method is repeated for different examples. The code remains the same except you have to determine the numerator and denominator which you can try as an exercise.
- The results of the different examples are as shown below and the GM and the PM are determined.

$$2)\ \ kG(s) = \frac{k}{s(s+1)(s+5)}$$

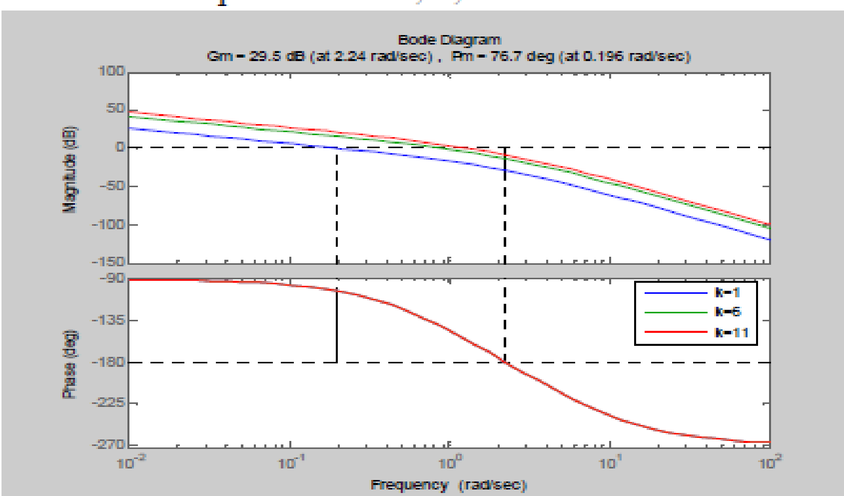
Root locus



Nyquist plots for k=1, 6, 11



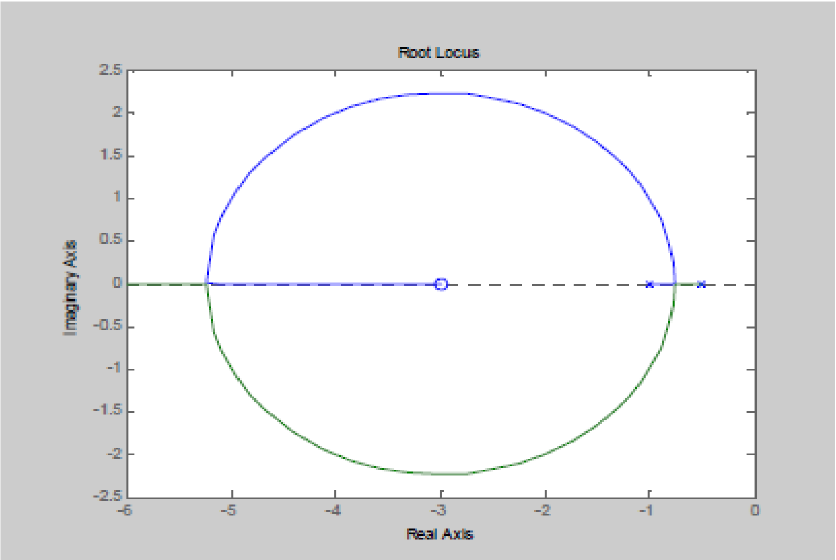
Bodeplots for k=1, 6, 11



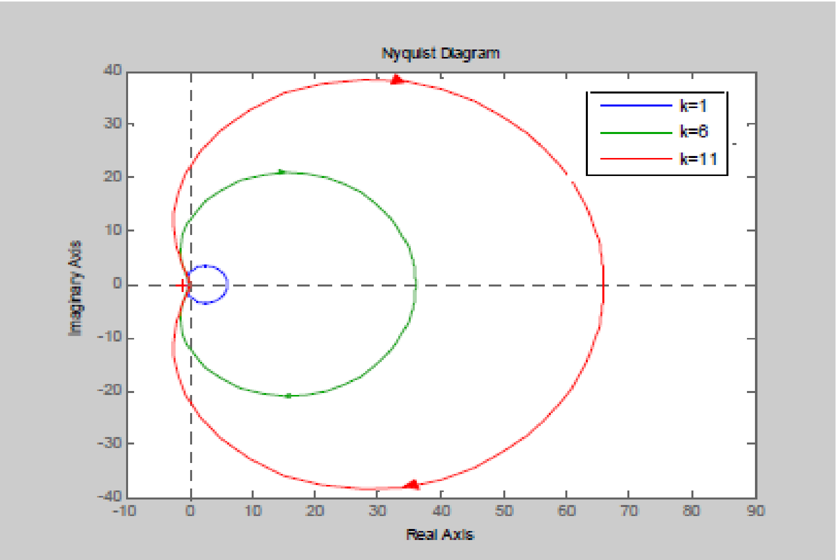
- For k = 1 the GM = 29.5dB and PM = 76.7deg.
- For k = 6 the GM = 14dB and PM = 38.5deg
- For k=11 the GM = 8.71dB and PM = 23deg
- The point at which the Nyquist plot crosses the x-axis can be determined using the Routh Hurwitz criterion: find the value of K, denoted by K1, that makes the system marginally stable. The auxiliary equation will then yield the value of w1, the frequency at which the system will oscillate if the gain is set to the value of K_1. Hence $1+ K_1G(j\omega_1)=0$, and the value of the Nyquist diagram at the -180^0 crossover is then $G(j\ \omega_1)=-1/\ K_1$. If the Nyquist diagram has multiple -180^o crossings, the Routh-Hurwitz criterion will indicate multiple values of K that force a row of zeros of the Routh array to zero and thus generate auxiliary equations. Each resulting value of K_i will give a different frequency ω_i. Note: the reason we can use the technique above is that K does not have any effect on the value of $G(j\omega)$. The gain K only magnifies or shrinks the Nyquist diagram without changing its shape.

5) $kG(s) = \dfrac{k(s+3)}{(s+1)(s+0.5)}$

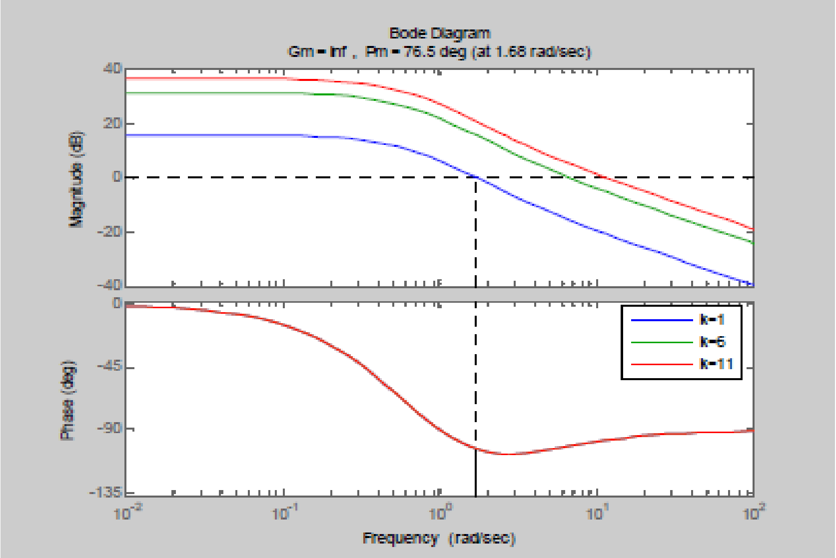
Root locus



Nyquist plots for k=1, 6, 11



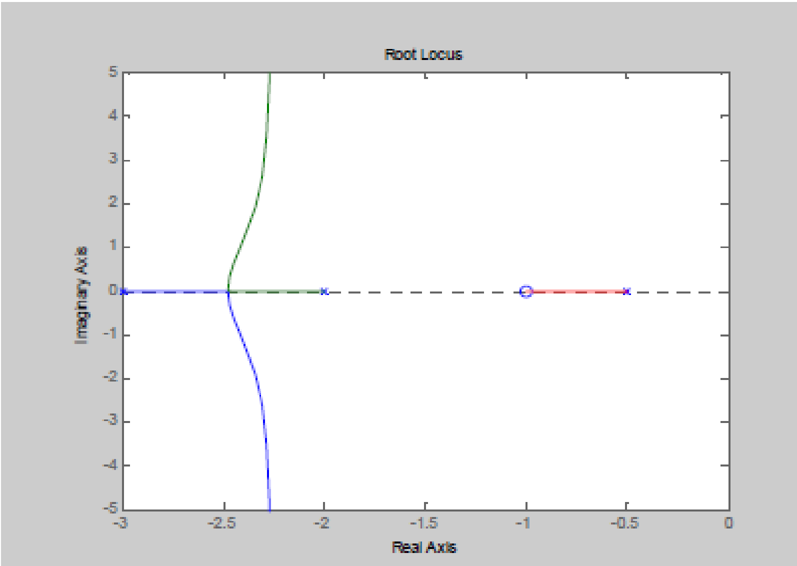
Bodeplots for k=1, 6, 11



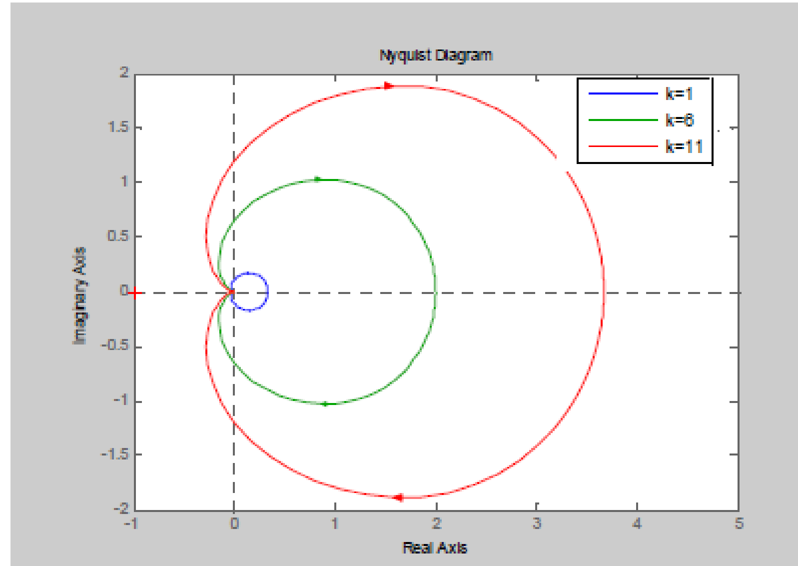
- For k=1 the GM = infinity and PM = 76.5deg.
- For k =6 the GM = infinity and the PM = 78.4deg
- For k=11 the GM = infinity and the PM = 82.7deg

6) $kG(s) = \dfrac{k(s+1)}{(s+0.5)(s+2)(s+3)}$

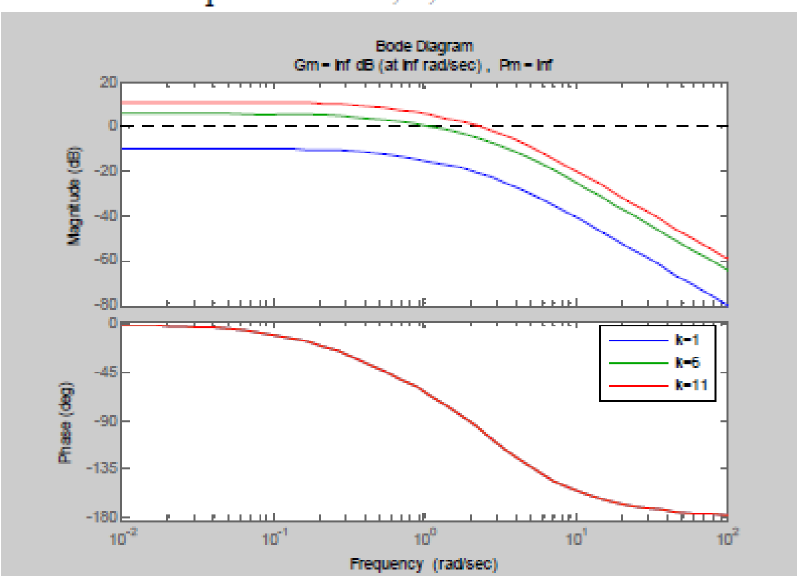
Root locus



Nyquist plots for k=1, 6, 11

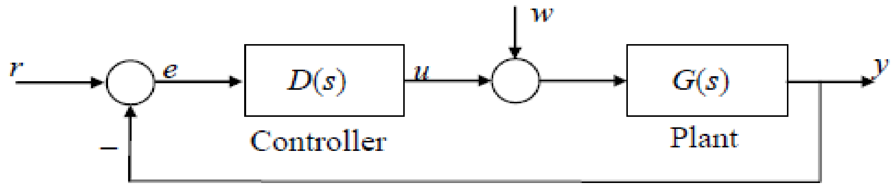


Bodeplots for k=1, 6, 11



- For k=1 the GM = infinity and PM = infinity.
- For k =6 the GM = infinity and the PM = 113deg
- For k=11 the GM = infinity and the PM = 81.5deg

UNITY FEEDBACK SYSTEMS REVISITED



RECALL -- for the Unity Feedback case, if *D(s)G(s)* has n free *s* in the denominator, then the system is of type *n*

e.g., D(s)G(s) = 1/[s(s+1)] is a type 1 system and will have a zero ss error for a step input and a constant steady state error for a ramp input.

<u>How do we calculate ss error for such systems</u>?

Note that for UFB systems only, the error constants are as follows:

position error constant $K_p = \lim_{s \to 0} D(s)G(s)$;

velocity error constant $K_v = \lim_{s \to 0} sD(s)G(s)$; **and**

acceleration error constant $K_a = \lim_{s \to 0} s^2 D(s)G(s)$ [*these terms are used in industry*]

TYPES? Type 0 – the system has a constant error to a step; Type 1 – system has a constant error to a ramp input,….and so on.

So, for type 0 systems, ss error to a step input $r(t) = 1(t)$ **is** $e_{ss} = \dfrac{1}{1+K_p}$

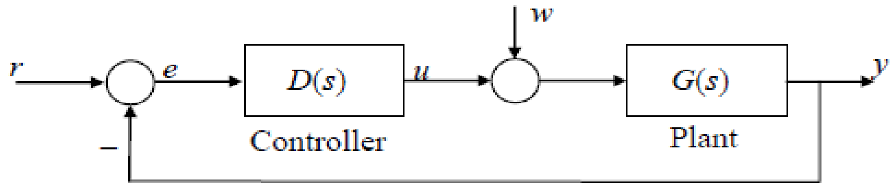
for type 1 systems, ss error to a ramp input $r(t) = t \cdot 1(t)$ **is** $e_{ss} = \dfrac{1}{K_v}$

for type 2 systems, ss error to parabolic input $r(t) = t^2 \cdot 1(t)$ **is** $e_{ss} = \dfrac{1}{K_a}$

<u>Find the ss error to the appropriate inputs for the following systems</u>:

H(s) = 1/(s+5); H(s) = 1/[s(s+7)]; H(s) = (s+4)/[s(s+10)]; H(s) = (s+2)/[s (s+3)(s+12)]

UNITY FEEDBACK SYSTEMS REVISITED



RECALL -- for the Unity Feedback case, if *D(s)G(s)* has n free *s* in the denominator, then the system is of type *n*

e.g., D(s)G(s) = 1/[s(s+1)] is a type 1 system and will have a zero ss error for a step input and a constant steady state error for a ramp input.

<u>How do we calculate ss error for such systems</u>?

Note that for UFB systems only, the error constants are as follows:

position error constant $K_p = \lim_{s \to 0} D(s)G(s)$;

velocity error constant $K_v = \lim_{s \to 0} sD(s)G(s)$; **and**

acceleration error constant $K_a = \lim_{s \to 0} s^2 D(s)G(s)$ [*these terms are used in industry*]

TYPES? Type 0 – the system has a constant error to a step; Type 1 – system has a constant error to a ramp input,….and so on.

So, for type 0 systems, ss error to a step input $r(t) = 1(t)$ **is** $e_{ss} = \dfrac{1}{1+K_p}$

for type 1 systems, ss error to a ramp input $r(t) = t \cdot 1(t)$ **is** $e_{ss} = \dfrac{1}{K_v}$

for type 2 systems, ss error to parabolic input $r(t) = t^2 \cdot 1(t)$ **is** $e_{ss} = \dfrac{1}{K_a}$

<u>Find the ss error to the appropriate inputs for the following systems</u>:

H(s) = 1/(s+5); H(s) = 1/[s(s+7)]; H(s) = (s+4)/[s(s+10)]; H(s) = (s+2)/[s (s+3)(s+12)]

6.8. COMPENSATOR DESIGN

Some addtl. design guidelines (to add to the %*OS*, t_r, t_s, ….and ss error formulae from before)

• For any stable minimum-phase system (that is, one with no RHP zeros or poles), the phase of G(jω) is uniquely related to the magnitude of G(jω)   (Bode's gain-phase theorem)
This implies that $\boxed{\angle G(j\omega) \approx n \times 90^0}$ where n is the slope of |G(jω)| in units of decade of amplitude per decade of frequency. For example, for a system with a first order term in the denominator (n=-1), the phase of the G(jω) curve will be -1 x 90^0 = -90^0.

• For stability, we want $\angle G(j\omega) > -180^0$ for PM >0. Therefore, using the above theorem, we can adjust the |K G(jω)| curve so that it has a slope of -1 at the 'crossover' frequency, ω_c (where |K G(jω)|=1). This will provide a PM of around 90^0. Typically, this design guideline is stated as follows:

Adjust the slope of the magnitude curve |K G(jω)| so that it crosses over magnitude 1 with a slope of -1 for a decade around ω_c.

• <u>Practice in industry</u>: GM should be > 9 dB, and the PM should be at least 40^0.

• Relationship between phase margin PM and damping ratio ζ:

$\zeta \approx PM/100$ where PM is in degrees

(this can be shown using the standard second order equation)

• Using the standard second order system equation, and plots of CL for difference phase margins, one can also conclude (see text for details) that

$$\omega_c \leq \omega_{BW} \leq 2\omega_c$$

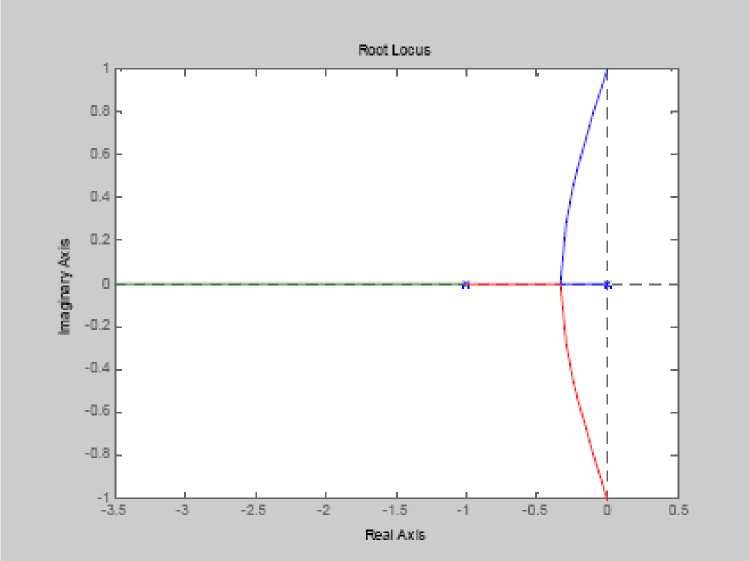
where ω_c is the cross over frequency (at which gain = 1.0), and ω_{BW} is the bandwidth of the system.

USING IDEAS FROM NYQUIST CRITERION FOR COMPENSATOR DESIGN

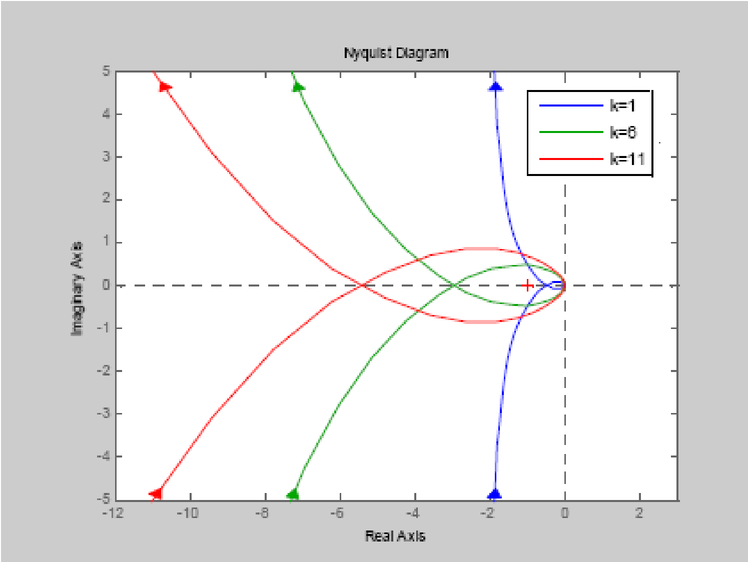
- RECALL that we don't want the Nyquist curve to encircle the (-1,0) point which is also called the 'critical point'. Also, recall that how far the curve is from that point determines the gain and phase margins. For instance, consider KG(s) = K/[s (s+1)(s+2)]. First draw the root locus for this system. Next draw the Nyquist and Bode plots for this system. Can you identify the GM and PM in all the plots? <u>Note: change in EITHER gain and phase changes (due to ??) can destabilize a system</u>!!
- We know how to perform compensator design using the root locus technique – all this does is tell us when the poles might go to the right half plane. It does not provide us with GM and PM information easily. How can we use a totally different technique, using frequency response plots (that are more common in industry) to perform compensator design?
- Looking at the Bode and Nyquist plots you just made, can you see how it could predict instability when the gain K is increased? (note that it is obvious from the root locus plot how this happens, but not so obvious in these plots initially – once you know it, it will be obvious from these plots too!)
- What are some guidelines you can come up with for compensator design based on the Nyquist criterion and some of the guidelines given earlier on this page?
- Sketch the Bode plots for lead and lag compensators and suggest how they can be used

$$4)\ kG(s) = \frac{k}{s(s+1)^2}$$

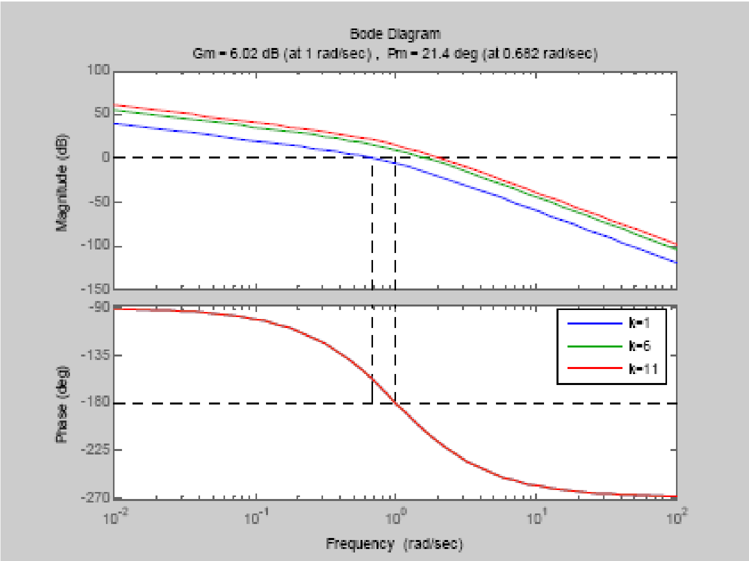
Root locus



Nyquist plots for k=1, 6, 11

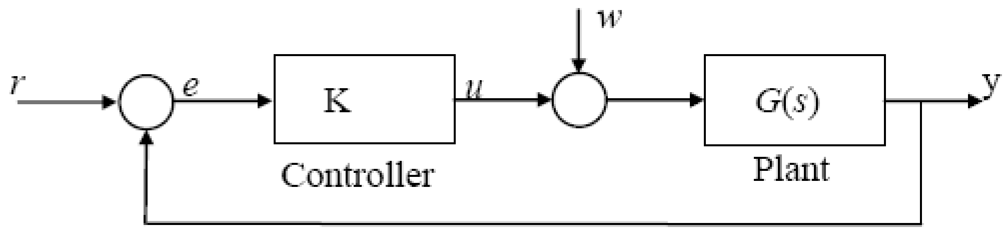


Bodeplots for k=1, 6, 11



- The nyquist plot shows that for the value k=1 the nyquist plot doesn't enclose (-1,0) but as the value of the k increases the plot encloses -1/k (since increase in k decreases 1/k).
- For k=1 the GM = 6.02dB and PM = 21.4deg.
- For k =6 the GM = -9.54dB and the PM = -27.1deg
- For k=11 the GM = -14.8dB and the PM = -38.5deg

DETERMINING STEADY STATE ERRORS USING A BODE PLOT



How do we calculate ss error using the Bode plot?

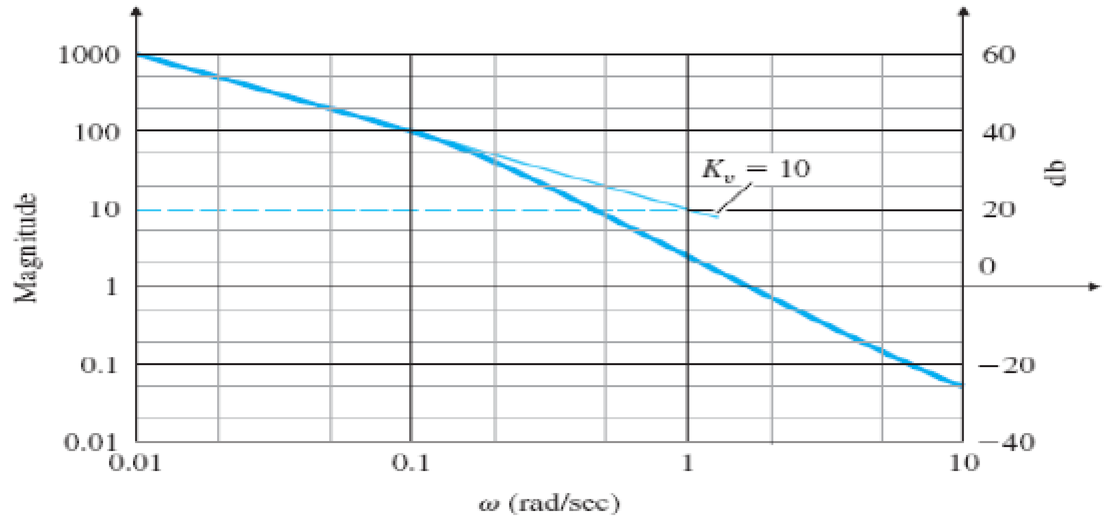
- **Recall from chapter 4 that the ss error for a system decreases as the OL gain increases**
- **Also, at very low frequencies, $KG(j\omega) = K_0(j\omega)^n$ (recall K_0 is the gain when the TF is cast in 'standard' form, and n is the number of free s terms in the denominator of the OLTF).**
  Conclusion – the larger the value of the magnitude of the low frequency asymptote, the lower the steady state errors will be for the closed loop system.
- **When n=0 (type 0 system), the low frequency asymptote is a constant and the gain Ko of the OL is equal to the position error constant K_p. Recall that $e_{ss} = \frac{1}{1+K_p}$ for a step input.**

When n=-1 (type 1 system), the low frequency slope has a value of -20 dB/decade, and this can be used to determine K_v directly from the plot. How? $K_v = \lim_{s\to 0} sD(s)G(s)$ = K_0. But how do we determine K_0 from the Bode plot? …… Remember that the low frequency gain is K_0/ω, and so on the magnitude plot, extend the low frequency asymptote to ω=1 rad/sec and read the value of the magnitude and that will be K_0. Alternately, one could read the magnitude at any frequency on the low-frequency asymptote and compute $K_0 = K_v = \omega A(\omega)$, where $A(\omega)$ is the magnitude of the low-frequency asymptote at the frequency ω.

Computation of K_v

Figure 6.13
Determination of K_v from the Bode plot for the system $KG(s) = 10/[s(s+1)]$

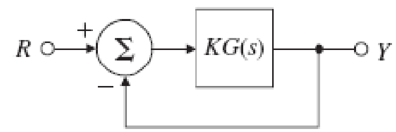
As an example of the determination of steady-state errors, a Bode magnitude plot of an open-loop system is shown in Fig. 6.13. Assuming that there is unity feedback as in Fig. 6.4, find the velocity-error constant, K_v.



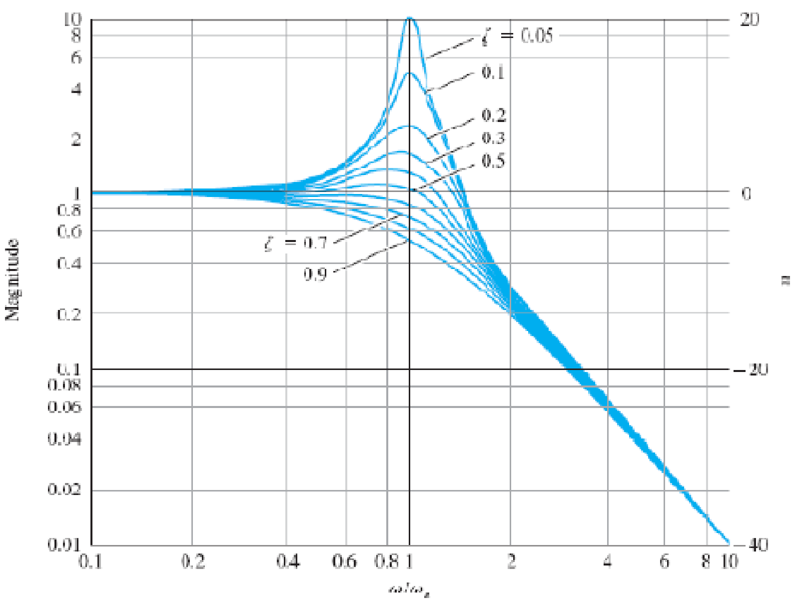
Solution. Because the slope at the low frequencies is −1, we know that the system is type 1. The extension of the low-frequency asymptote crosses $\omega = 1$ rad/sec at a magnitude of 10. Therefore, $K_v = 10$ and the steady-state error to a unit ramp for a unity feedback system would be 0.1. Alternatively, at $\omega = 0.01$ we have $|A(\omega)| = 1000$; therefore, from Eq. (6.23) we have

$$K_o = K_v \cong \omega|A(\omega)| = 0.01(1000) = 10.$$

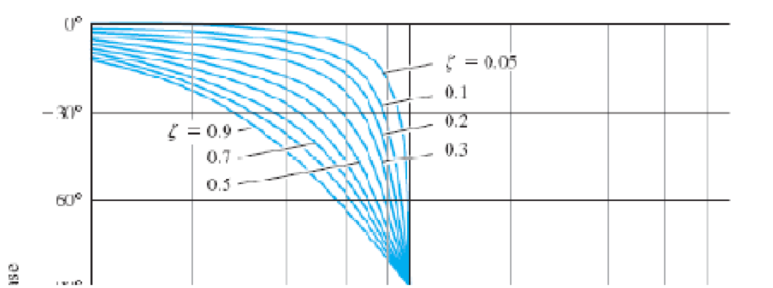
Design equations related to system bandwidth

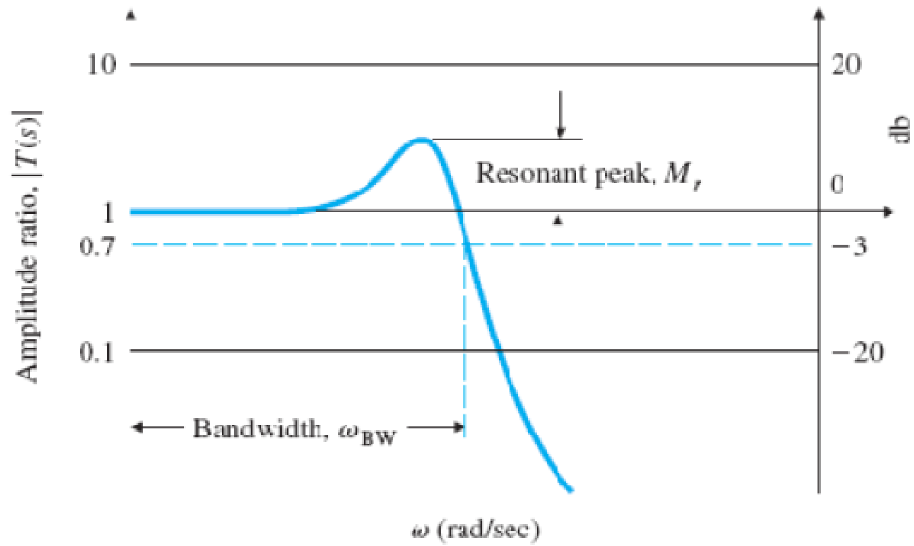


• **Bandwidth** is defined to be the maximum frequency at which the output of a system will track an input sinusoid in a satisfactory manner. By convention, for the system shown alongside with a sinusoidal input r , the bandwidth is the frequency of r at which the output y is attenuated to a factor of 0.707 times the input.



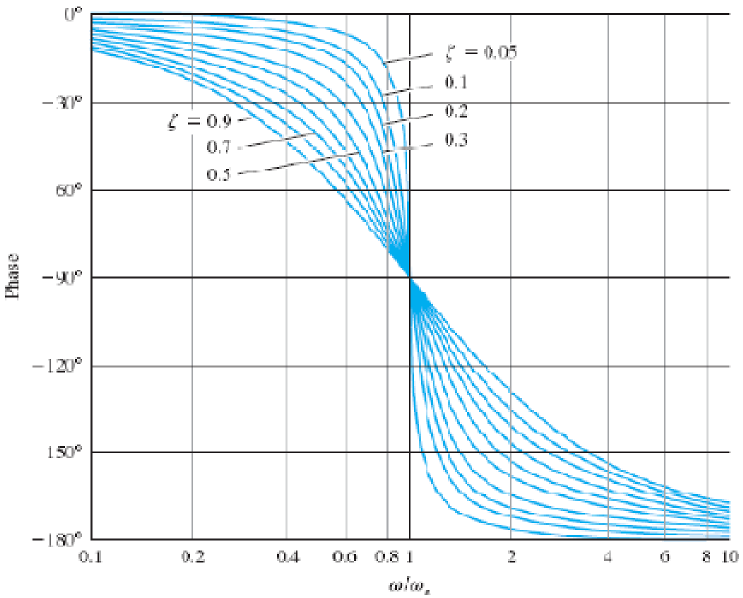
(a)





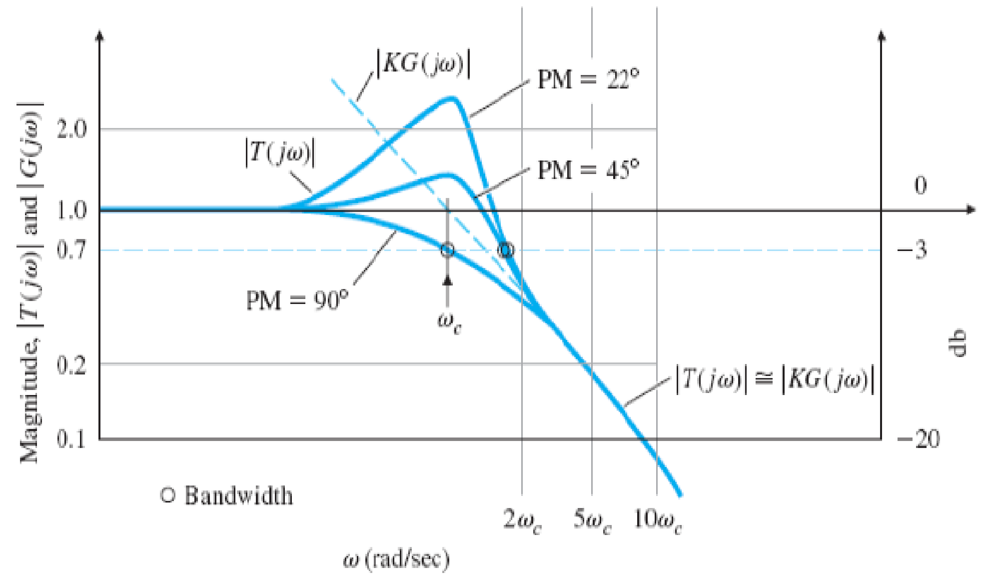
• Bandwidth - a measure of speed of response, similar to rise time and peak time. In fact, we can see that the bandwidth will equal the natural frequency of the closed-loop root (that is, $\omega_{BW} = \omega_n$ for a closed-loop damping ratio of $\zeta = 0.7$). For other damping ratios, the bandwidth is approximately equal to the natural frequency of the closed-loop roots, with an error typically less than a factor of 2.

(a)



(b)

• Bandwidth - a measure of speed of response, similar to rise time and peak time. In fact, we can see that the bandwidth will equal the natural frequency of the closed-loop root (that is, $\omega_{\mathrm{BW}} = \omega_n$ for a closed-loop damping ratio of $\zeta = 0.7$). For other damping ratios, the bandwidth is approximately equal to the natural frequency of the closed-loop roots, with an error typically less than a factor of 2.



• Consider a system where in which $|KG(j\omega)|$ show the typical behavior,

$$|KG(j\omega)| \gg 1 \quad \text{for} \quad \omega \ll \omega_c,$$

$$|KG(j\omega)| \ll 1 \quad \text{for} \quad \omega \gg \omega_c,$$

The closed loop frequency response can then be approximated by

$$|\mathcal{T}(j\omega)| = \left|\frac{KG(j\omega)}{1+KG(j\omega)}\right| \cong \begin{cases} 1, & \omega \ll \omega_c, \\ |KG|, & \omega \gg \omega_c. \end{cases} \qquad (6.36)$$

• …..The approximations in Eq. (6.36) were used to generate the curves of $|T(j\omega)|$. It shows that the bandwidth for smaller values of PM is typically somewhat greater than ω_c , though usually it is less than $2\omega_c$; thus $\omega_c \le \omega_{\mathrm{BW}} \le 2\omega_c$

Sketch the Bode plot for a lead compensator D(s) = $D(s) = \dfrac{Ts+1}{\alpha Ts+1}$ **= (0.1057 s + 1)/(0.01952 s + 1)**

(i) Find T and α
(ii) Then sketch the Bode plots

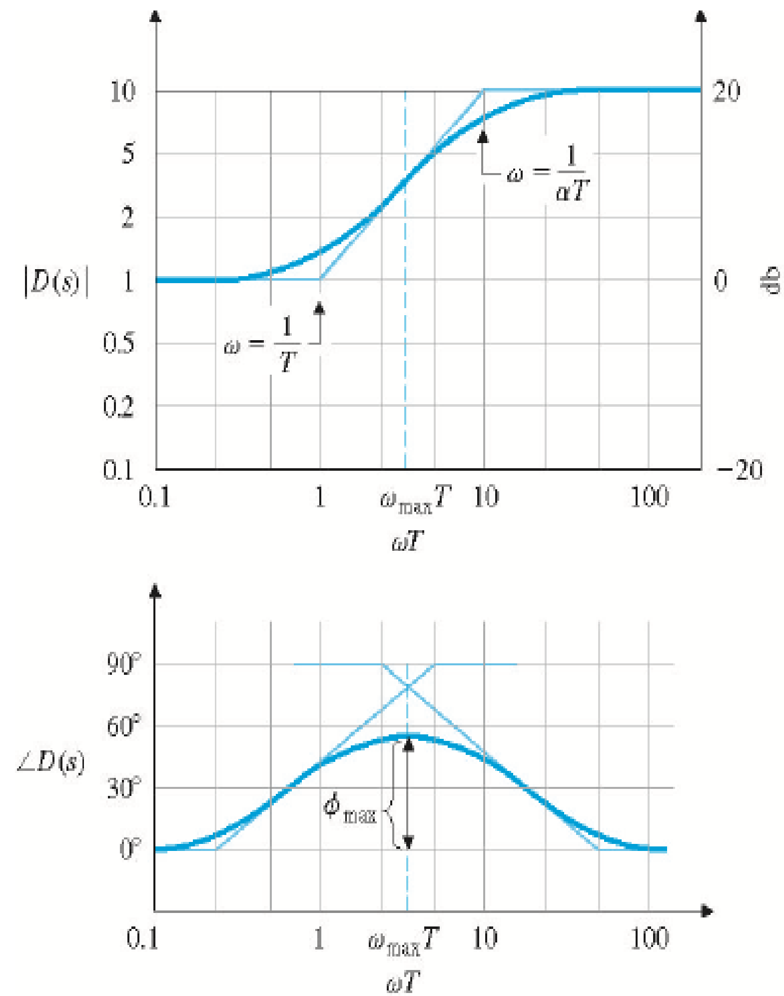
Now sketch the Bode plot for a lag compensator function $D(s) = \alpha \dfrac{Ts+1}{\alpha Ts+1}$ **= (60s + 15)/(60s + 1)**

$\alpha > 1$ **(<u>note</u>: an addtl. gain** α **is multiplied in this lag case, compared to the lead case. The controller will also include another overall gain term** ***K*** **as in the lead case.)**
Similar to what you did above, find T and α, and then the Bode plots**.**

LEAD COMPENSATOR

- The form of the lead compensator in the text is the transfer function $D(s) = \dfrac{Ts+1}{\alpha Ts+1}$ with $\alpha < 1$, where $1/\alpha$ is the ratio of th

pole location/zero location, i.e., $\alpha = (1/T)/(1/\alpha T)$.



- The phase contributed by the lead compensation in Eq. (6.38) is given by

$$\boxed{\varphi = \tan^{-1}(T\omega) - \tan^{-1}(\alpha T\omega)}$$

- It can be shown (see Problem 6.43) that the frequency at which the phase is maximum is given by $\boxed{\omega_{max} = 1/T\sqrt{\alpha}}$

The maximum phase contribution, that is, the peak of the *D(s)* curve in the figure alongside corresponds to

$$\boxed{\sin\varphi_{max} = (1-\alpha)/(1+\alpha)} \quad \text{or}$$

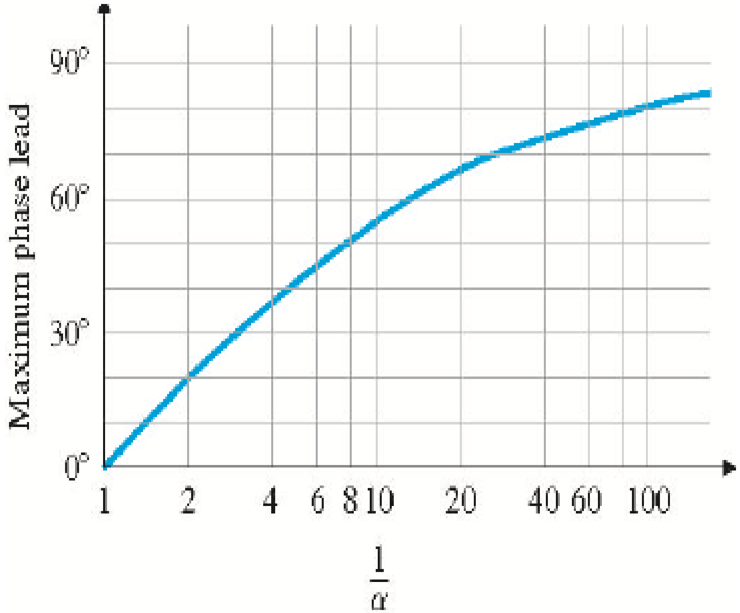
$$\boxed{\alpha = (1-\sin\varphi_{max})/(1+\sin\varphi_{max})}$$

- Another way to look at this is that the maximum phase occurs at a frequency that lies midway between the two break-point frequencies (sometimes called corner frequencies) on a logarithmic scale – check in figure on the right.

- Note that the amount of phase lead at the midpoint depends only on α since

$$\sin \varphi_{max} = (1 - \alpha)/(1 + \alpha)$$

and that relation is captured in the figure alongside



- Higher values of 1/a (see gain plot) also produces amplification at higher frequencies. Thus our task is to select a value of a that is a good compromise between an acceptable PM and an acceptable noise sensitivity at high frequencies. Usually the compromise suggests that a lead compensation should contribute a maximum of 70^0 to the phase. If a greater phase lead is needed, then use a double-lead compensator.

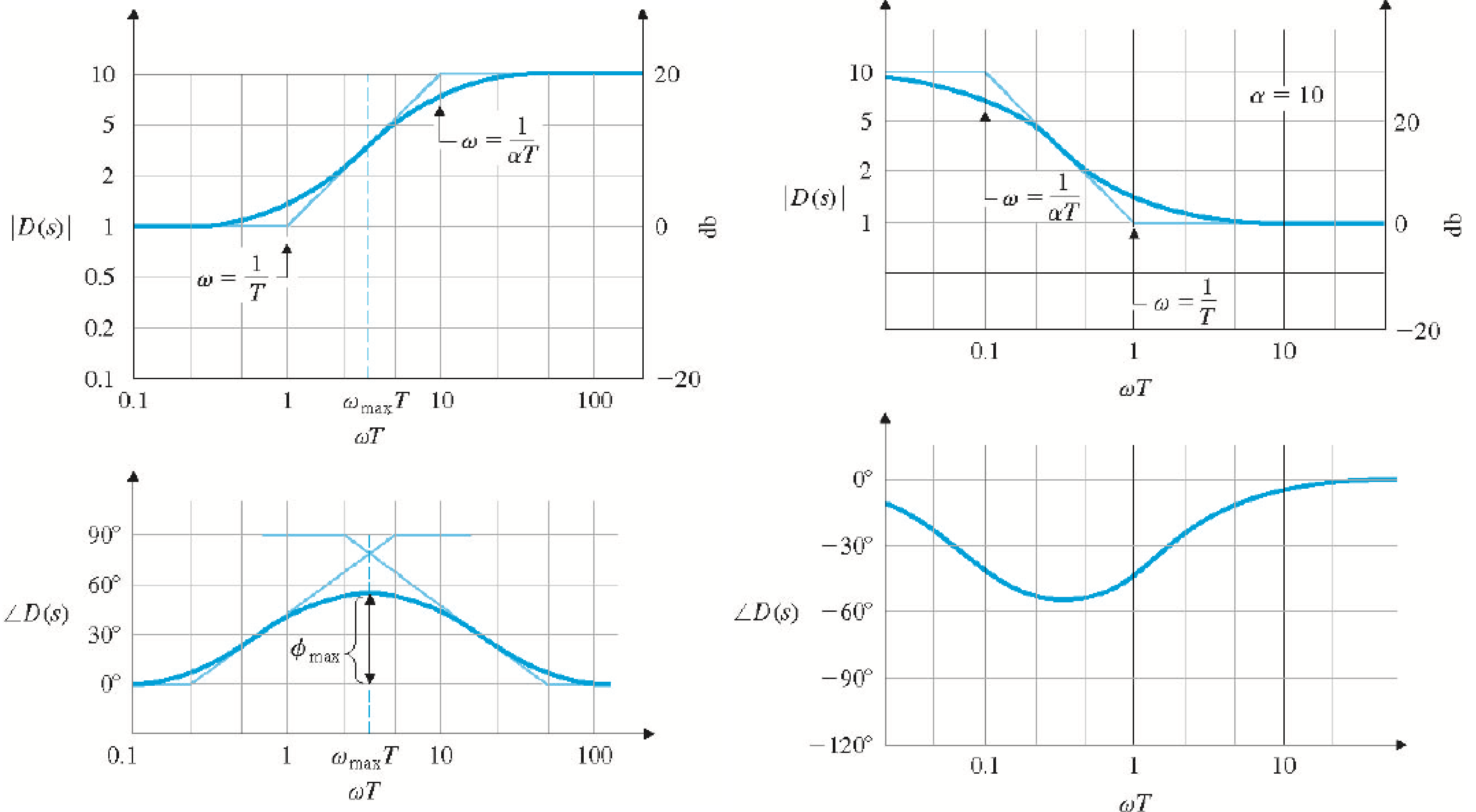
- In pole zero form, D(s) = (s+z)/(s+p), it can be shown (see pb. 6.43) that $\omega_{max} = \sqrt{|z||p|}$

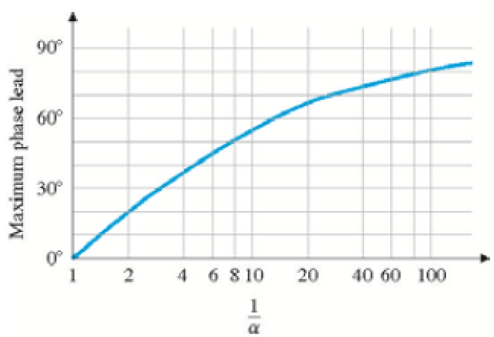
- It is also easy to show (can you?) that the compensator gain at ω_{max} is $1/\sqrt{\alpha}$, i.e., $|D(j\omega_{max})| = 1/\sqrt{\alpha}$

- All the relations above hold for the lag compensator also – can you check and convince yourself?

6.9. COMPENSATOR DESIGN PROCEDURE

Form of the compensator: $D(s) = \dfrac{Ts+1}{\alpha Ts+1}$ with α<1 for lead and $D(s) = \alpha\dfrac{Ts+1}{\alpha Ts+1}$ with α>1 for lag compensators (note: the extra α for the lag compensator; also note that gain K is considered separately in this design). What does its Bode plot look like? Remember that when you multiply G(s) with D(s), they ADD in Bode plots! Let us study some of the characteristics of this compensator. Consider the lead case. Note that it 'adds' both gain and phase to the G(s) plots. At what frequency is the 'extra' phase that is being added at its maximum? That has to be determined so that we can place the pole/zero of this compensator such that this frequency happens to be at the crossover point on the gain plot to get maximum phase margin – that is the key issue. The magnitude of the maximum phase (ϕ_{max}) that can be added for the compensator is solely a function of α - a little thought will clarify why. The derivation uses high school math, and you will find that α and ϕ_{max} are related by the following equation: $\sin\phi_{max} = (1-\alpha)/(1+\alpha)$. So, if we desire a certain ϕ_{max}, one can compute the corresponding α (remember this is the ratio of the zero to the pole) that will get us this phase addition, from the equation $\alpha = (1-\sin\phi_{max})/(1+\sin\phi_{max})$. The phase contributed by the lead system is given by $\phi = \tan^{-1}(Tw) - \tan^{-1}(\alpha Tw)$. The frequency at which the phase is maximum is given by $w_{max} = 1/T\sqrt{\alpha}$ which is the frequency mid-way between the pole and zero on a logarithmic scale.





Figures. Bode plots of lead (top left) and lag (top right) compensators, with $\alpha = 10$. Note that this just gets 'added' to the Bode plots of the OLTF. One of the challenges is to how to select α. The figure on the left shows how phase margin changes with α.

NOTE:

- **To reduce steady state error only, you have to increase the <u>low-frequency gain</u> without affecting the phase margin (i.e., don't change the gain cross over frequency). How will you do that? (LAG COMPENSATION)**
- **To improve transient features only, you will have to change the PM and gain cross over without affecting ss error. How will you do that? (LEAD COMPENSATION)**

DESIGN PROCEDURE FOR LEAD COMPENSATION

Consider the lead compensation transfer function $D(s) = \dfrac{Ts+1}{\alpha Ts+1}$ $\alpha<1$. Then the design procedure for this lead compensation is as follows:

1. Determine open-loop gain K to satisfy error or bandwidth requirements:

(a) to meet error requirement, pick K to satisfy error constants (K_p , K_v ,or K_a) so that e_{ss} error specification is met, or alternatively,

(b) to meet bandwidth requirement, pick K so that the open-loop crossover frequency is a factor of two below the desired closed-loop bandwidth.

2. Evaluate the phase margin (PM) of the uncompensated system using the value of K obtained from Step 1.

3. Allow for extra margin (about 10°), and determine the needed phase lead φ_{max}.

4. Determine α using $\boxed{\alpha = (1 - \sin\varphi_{max})/(1 + \sin\varphi_{max})}$

5. Pick ω_{max} to be at the crossover frequency on the Bode plot (of OLTF). Thus the compensator zero is then at $1/T = \omega_{max}\sqrt{\alpha}$ and the compensator pole is at $1/\alpha T = \omega_{max}/\sqrt{\alpha}$.

6. Draw the compensated frequency response and check the PM.

7. Iterate on the design. Adjust compensator parameters (poles, zeros, and gain) until all specifications are met. Add an additional lead compensator (that is, a double lead compensation) if necessary, i.e., if the lead required is more than 70^0.

DESIGN PROCEDURE FOR LAG COMPENSATION

Consider the lag compensation transfer function $D(s) = \alpha \dfrac{Ts+1}{\alpha Ts+1}$ $\alpha>1$ (note: an addtl. gain α is multiplied in this lag case, compared to the lead case. The controller will also include another overall gain term K as in the lead case.)

1. Determine the open-loop gain K that will meet the phase-margin requirement without dynamic compensation (note: this will adjust the gain of the system, but not for steady state error correction but only for PM requirement!)
2. Draw the Bode plot of the uncompensated system with crossover frequency from Step 1, and evaluate the low-frequency gain.
3. Determine α to meet the low-frequency gain error requirement, i.e., steady state error requirement.
4. Choose the corner frequency $\omega = 1/T$ (the zero of the lag compensator) to be one octave to one decade below the new crossover frequency ω_c (note: ω_c is the gain that corresponds to the phase margin location; in the MATLAB command [Gm,Pm,Wcg,Wcp]=margin(g), Wcg corresponds to the location of Gm, and Wcp corresponds to the location of Pm).
5. The other corner frequency (the pole location of the lag compensator) is then $\omega = 1/\alpha T$.
6. Iterate on the design. Adjust compensator parameters (poles, zeros, and gain) to meet all the specifications.

Sketch the root locus, Bode plot and Nyquist plot for a stable system that does not have the required transient and steady state features.

Try two systems (i) a second order system (e.g., $G(s)=1/[(s+1)(s+2)]$); and (ii) a third order system (you do this one on your own outside class).

(i) Can you state the control problem for the system, starting with customer specs?

(ii) Now design a compensator to achieve the customer specs using the root locus method,

(iii) Now perform the compensator design using Bode plots to see the 'parallel' between the two methods – do this using both lead and lag designs

Challenge question - Sketch the root locus, Bode plot and Nyquist plot for an unstable system starting with a TF model and suggest how you might stabilize it using a 'compensator' designed in the Bode plot domain.
(you do this outside class)

Now solve the problem using MATLAB
– can you write the code to automate the procedure you just developed?

6.10. LEAD COMPENSATOR DESIGN USING MATLAB (Pb. 6.49 text)

1. Problem statement. The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(1+s/5)(1+s/200)}$$

. Design a lead compensator for G(s) so that the CL system specs are:

- steady-state error to a unit ramp input is less than 0.01.
- the dominant closed-loop poles have a damping ratio $\xi \geq 0.4$.

2. Design procedure

(i) Consider steady-state error requirement to determine open-loop gain K.

The steady-state error to a unit ramp input is less than 0.01 means that $K_v \geq 1/0.01$

where $K_v = \lim_{s \to 0} sKG(s) = K$ **, thus we need $K \geq 100$.**

(ii) Transient requirements: Require $\zeta \geq 0.4 \Leftrightarrow PM \geq 40^0$

For $G(s) = \frac{K}{s(1+s/5)(1+s/200)}$ **, K=100, draw bode plot and evaluate the phase margin Pm.**

MATLAB code:

```
k=100;  num=k;
den1=tf([1 0],[1]); den2=tf([0.2 1],[1]); den3=tf([0.005 1],[1]);
den=den1*den2*den3;  g=num/den;
margin(g)
[Gm,Pm,Wcg,Wcp]=margin(g);
```

For the design, find gain margin (at Wcg), phase margin (at Wcp)

```
» Gm,Pm,Wcg,Wcp
Gm =
   2.0500
Pm =
   6.5144
Wcg =
  31.6228
Wcp =
  22.0149
```

- **We need to obviously increase phase margin. How? Need to find out how much phase to add, and where to select the zero and pole. To do this, first compute the maximum phase angle ϕ_{max} of the compensator.**

Require $\zeta \geq 0.4 \Leftrightarrow PM \geq 40^0$. Note that the cross over frequency will also change, and so we need some 'buffer'. Let that buffer be 10^0. So, we need to add, the difference between what the system already has, and then add this buffer.

```
PM=40;Q=PM-Pm+10;
```

then type:

```
»Q
Q =
  43.4856
```

so, we get ϕ_{max}=44^0.

- **Then compute α to get this ϕ_{max} (remember that they are related by Eqn. 6.40 of text),**

```
a=(1-sin(Q*pi/180))/(1+sin(Q*pi/180));
```

```
» a
a =
   0.1847
```

- **Now pick the location for ω_{max} of the compensator – at the gain cross over frequency**

```
Wmax=Wcp;
T=1/(Wmax*sqrt(a));
numds=[T 1]; dends=[a*T 1];
ds=tf(numds,dends); figure(2); bode(ds)
```

This gives T and the compensator D(s) as follows:

```
» Wmax
Wmax =
  22.0149
» T
T =
   0.1057
» ds
Transfer function:
0.1057 s + 1
-------------
0.01952 s + 1
```

- **For D(s) and G(s), with k=100 (note: D(s) does not affect ss error since it does not affect K_v, i.e., adds nothing at zero frequency)**

```
gds=g*ds
figure(1); hold on; margin(gds)
[Gm1,Pm1,Wcg1,Wcp1]=margin(gds);
```

Then type:

```
» Gm1,Pm1,Wcg1,Wcp1
Gm1 =
   4.2262
Pm1 =
  33.6024
Wcg1 =
  95.5482
Wcp1 =
  41.1178
```

The phase margin has not improved enough. So, have to iterate …………………

ITERATION PROCESS: Move compensator ω_{max} to the right by 10 rad/sec, because……recall that the compensator adds gain also and will move Wcp to the right. So, add 10 rad/sec to Wcp, and try that out (another option would be to increase the 'buffer' of phase that was added – you can try that out yourself separately)

The phase margin has not improved enough. So, have to iterate …………………

ITERATION PROCESS: Move compensator ω_{max} to the right by 10 rad/sec, because……recall that the compensator adds gain also and will move Wcp to the right. So, add 10 rad/sec to Wcp, and try that out (another option would be to increase the 'buffer' of phase that was added – you can try that out yourself separately)

```
Wmax=Wcp+10;
T=1/(Wmax*sqrt(a));
numds=[T 1]; dends=[a*T 1];
ds=tf(numds,dends); figure(2); bode(ds)
```

```
gds=g*ds
figure(1); hold on; margin(gds)
[Gm2,Pm2,Wcg2,Wcp2]=margin(gds);
```

```
» Gm2,Pm2,Wcg2,Wcp2
Gm2 =
    6.3192
Pm2 =
   41.8121
Wcg2 =
  111.8344
```

Wcp2 =
34.5915
» ds
Transfer function:

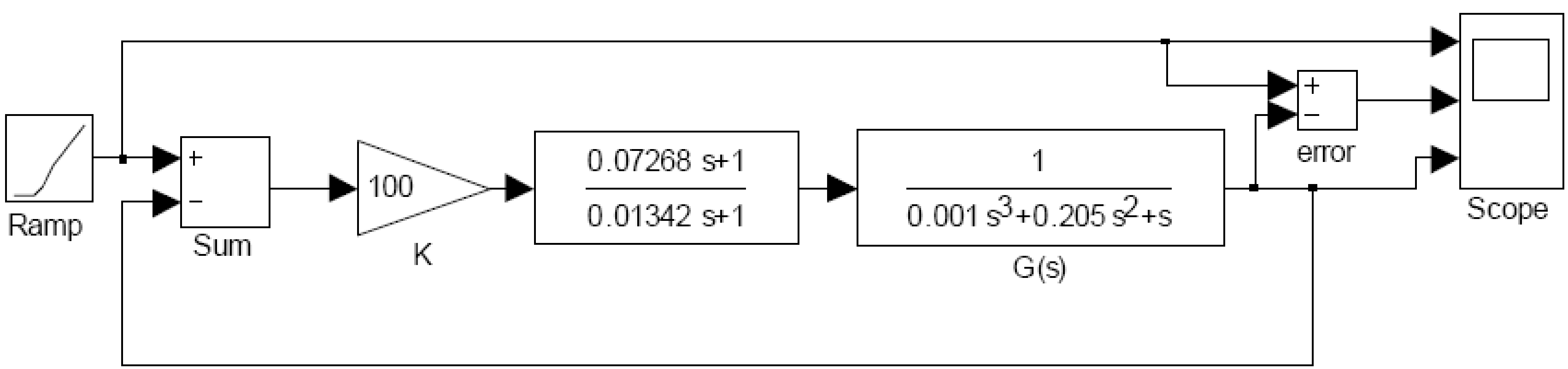
```
0.07268 s + 1
-------------
0.01342 s + 1
```

» gds

Transfer function:

```
                7.268 s + 100
---------------------------------------------
1.342e-005 s^4 + 0.003752 s^3 + 0.2184 s^2 + s
```

- **You will find that this does satisfy the design specifications - see Simulink structure below. Try also ltiview ({'step';'nyquist';'bode'},gds)**



6.11. LAG COMPENSATOR DESIGN USING MATLAB (Pb. 6.50 in text)

1. Problem Statement

The open-loop transfer function of a unity feedback system is

$$G(s) = \frac{K}{s(1+s/5)(1+s/50)}$$

Design a lag compensator for G(s) so that the closed-loop system satisfies the following specifications

- The steady-state error to a unit ramp input is less than 0.01.
- PM $\geq$ 40 degrees. Basically, we are using the same specs as in the previour problem, but designing a lag compensator.

2. Design procedure

Steps

(1) For $G(s) = \dfrac{Kg}{s(1+s/5)(1+s/50)}$, **choose Kg, draw bode plot and evaluate the phase margin Pm. You will find that it will be too large for low Kg and so Kg can be increased. Iterate till you get PM close to 40^0.**

```
Kg=1;
g1=tf([1],[1 0]); g2=tf([1],[1/5 1]); g3=tf([1],[1/50 1]);
g=Kg*g1*g2*g3;
margin(g)
[Gm,Pm,Wcg,Wcp]=margin(g);
Pm=Pm
```

Try different Kg and you can find that with Kg=7, the Pm is 39.6 degrees

(2) Use the MATLAB code above, with Kg=7 in $G(s) = \dfrac{Kg}{s(1+s/5)(1+s/50)}$

and then type:

```
» Wcp=Wcp
Wcp =
    4.9492
```

So we get the cross over frequency Wcp=4.95 rad/s. (can use ltiview also)

(3) Consider steady-state error requirement to determine the overall open-loop TF gain K.

The steady-state error to a unit ramp input is less than 0.01 means that $K_v \geq 1/0.01$**,**

where $K_v = \lim_{s \to 0} sKG(s) = K$ **. This implies that K>= 100.**

Thus, α=K/Kg=100/7=14.2857, and so you can set α=15.

(4) Choose corner frequency $W_2=1/T$ **(typically at least one decade below Wcp)**

W_2**=Wcp/20=0.25 rad/sec. To get this, we have to pick T=4.**

(5) Then the other corner frequency can be solved as follows:

$W_1=1/(T^* \alpha)=1/60$ **rad/s, thus Tα=60.**

We now know the parameters of the compensator - α and T.

(6) Draw the bode plot for D(s)G(s) to check phase margin.

```
a=15;
T=4;
ds=tf([a*T a],[a*T 1]);
dgs=ds*g;
hold on;
margin(dgs)
[Gm2 Pm2 Wcg2 Wcp2]=margin(dgs);
```

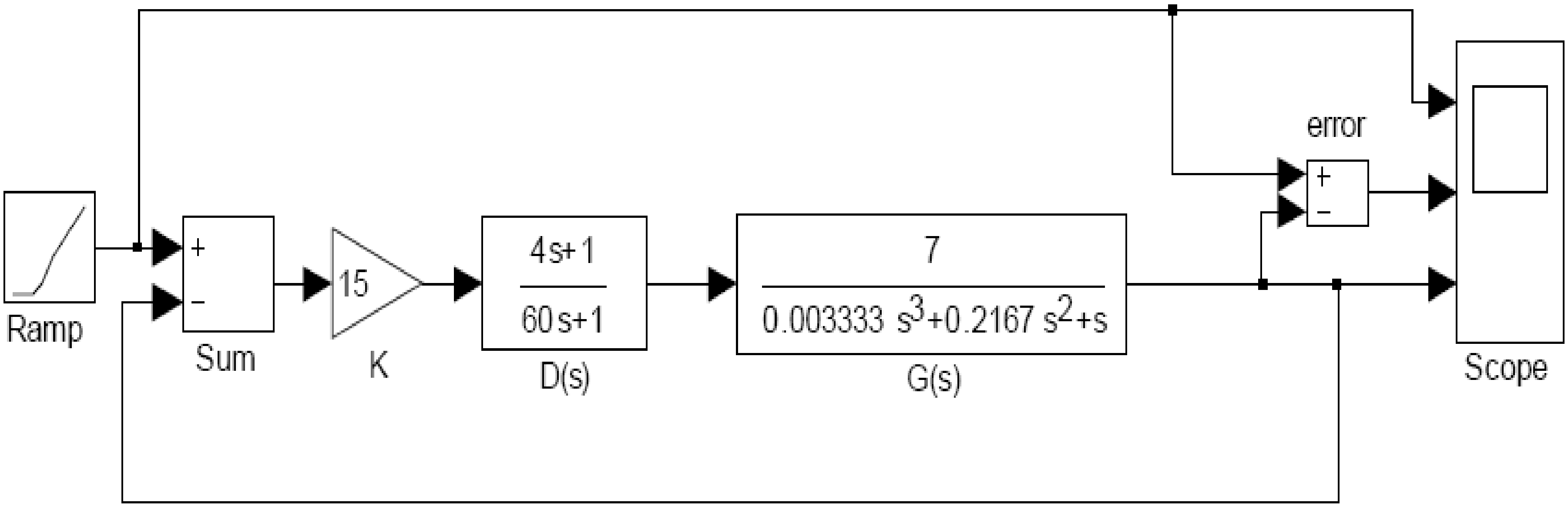
```
» ds
 Transfer function:
60 s + 15
------------
60 s + 1
```

Then type:

```
» PM=Pm2
PM =
   36.9077
```

So PM =36.9^{0}, which can revised again in case it is not close enough to the design requirement.

(7). Use Simulink to simulate ramp response for validation. Try also ltiview ({'step';'nyquist';'bode'},dgs)



SEE TEXT FOR LOTS OF EXAMPLES OF LEAD & LAG DESIGNS – Examples 6.14 – 6.19. OMIT ALL SECTIONS FROM 6.7.6 ONWARDS

- To study frequency response, we use two types of graphical representations

1. The Bode Plot:

   Plot of $20\log_{10}(AR)$ (in dB)vs. ω on loglog scale

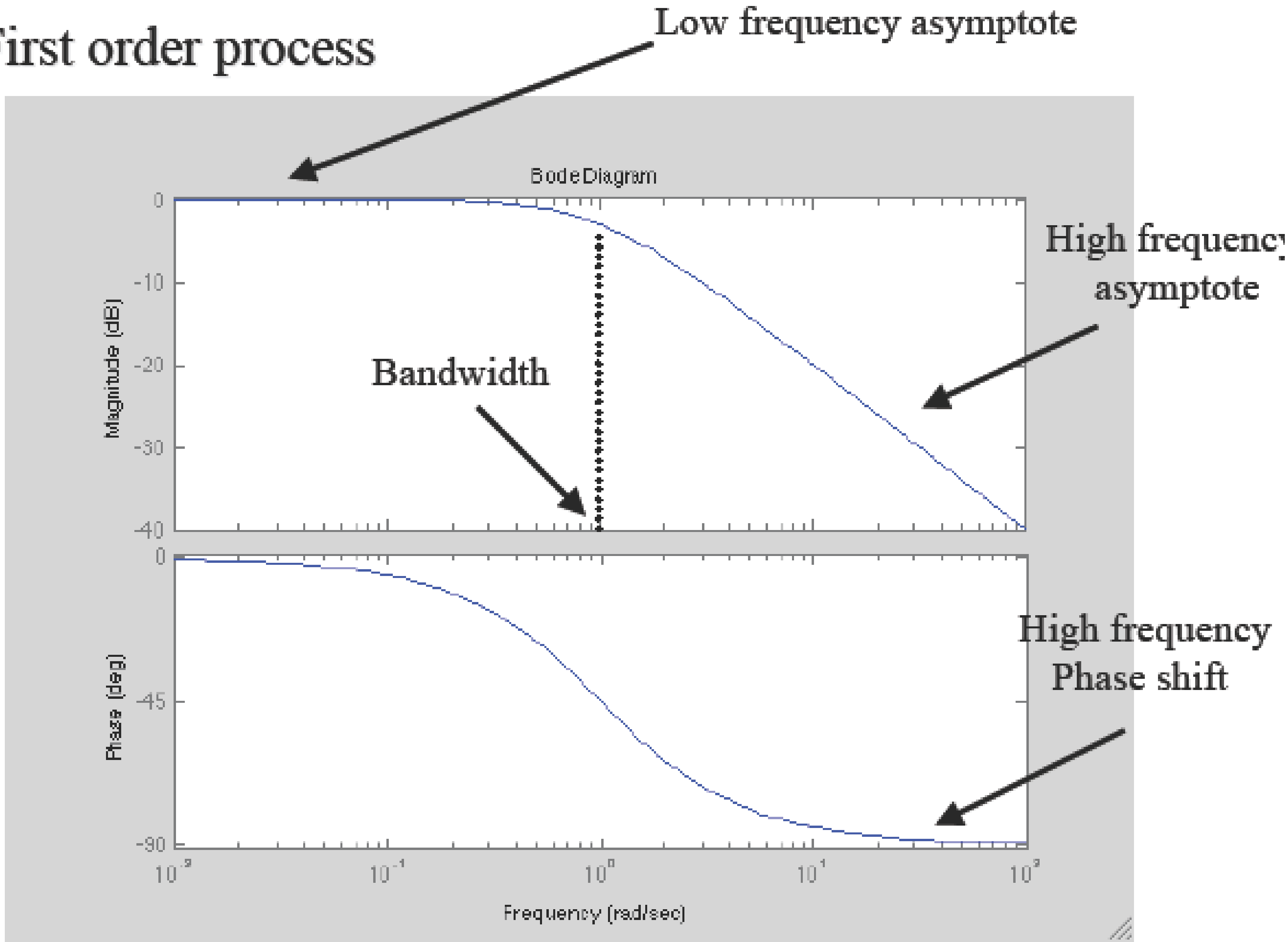
   Plot of ϕ (in degrees) vs. ω on semilog scale

2. The Nyquist Plot:

   Plot of the trace of $G(j\omega)$ in the complex plane

- Plots lead to effective stability criteria and frequency-based design methods

- First order process



Characteristic of the Bode plot

- low frequency asymptote

$$\lim_{\omega \to 0} \frac{K}{\sqrt{\tau^2\omega^2+1}} = K$$

- High frequency asymptote (if $\omega >>> 1/\tau$)

$$\frac{K}{\sqrt{\tau^2\omega^2+1}} \approx \frac{K}{\tau\omega}$$

  - Yields a line on loglog scale,

$$\log\left(\frac{K}{\tau\omega}\right) = \log K - \log(\tau) - \log(\omega)$$

  - The phase shift at high frequencies

$$\lim_{\omega \to \infty} -\tan^{-1}(\tau\omega) = -\frac{\pi}{2} = -90 \text{ deg}$$

- Characteristics of the Bode plot:
  - *Cut-off frequency* - this is the frequency ω_c at which

$$AR = \frac{AR(0)}{\sqrt{2}}$$

  - For first order systems

$$\frac{K}{\sqrt{\tau^2\omega_c^2+1}} = \frac{K}{\sqrt{2}} \Rightarrow \omega_c = \frac{1}{\tau}$$

  - The cut-off frequency is also the *bandwidth* of the first order system

- First order unstable system

$$G(s) = \frac{K}{\tau s - 1}$$

- Frequency response

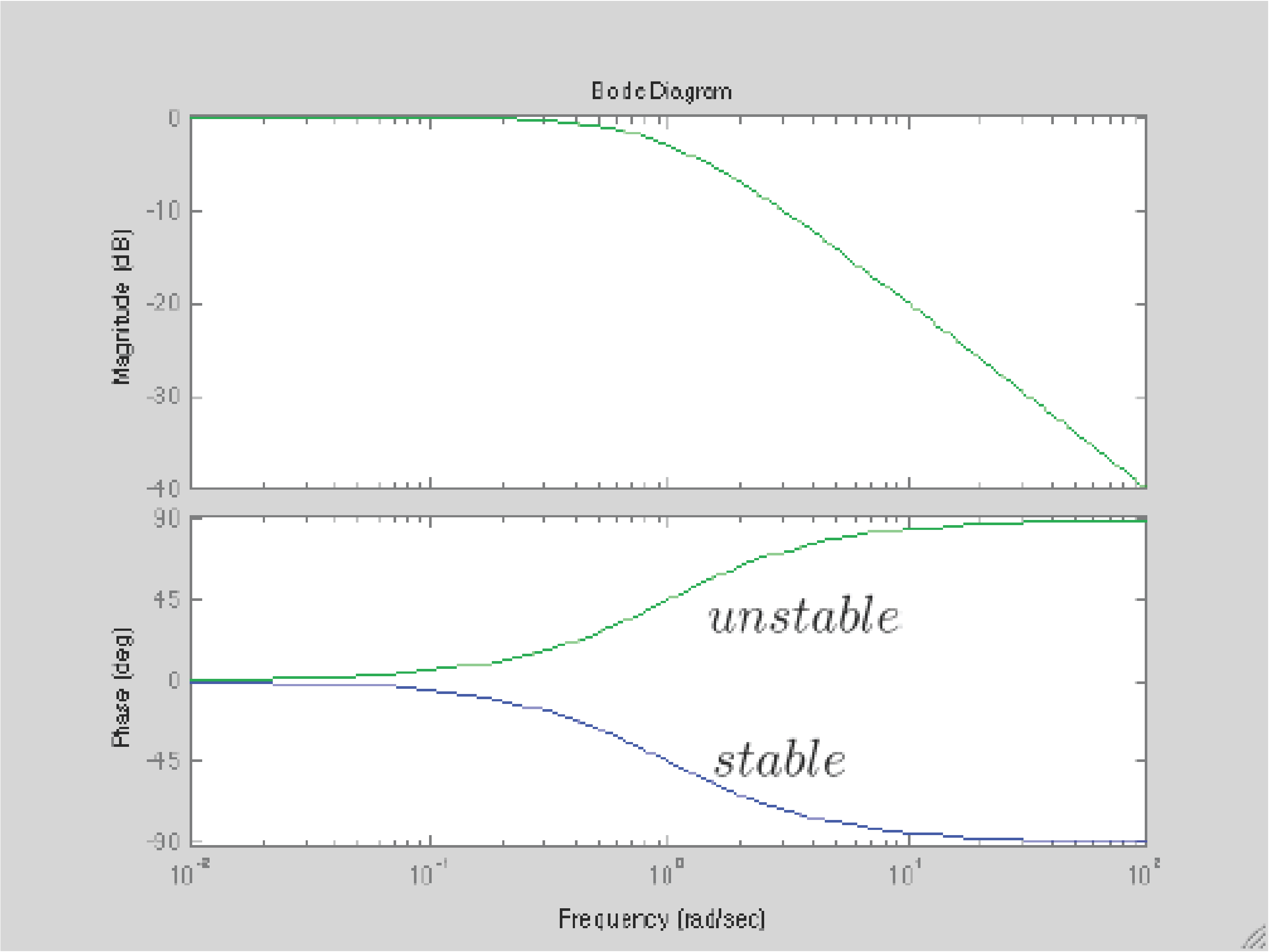
$$G(j\omega) = \frac{K}{\tau\omega j - 1}\frac{1+\tau\omega j}{1+\tau\omega j} = \frac{K + K\tau\omega j}{1+\tau^2\omega^2}$$

- Amplitude ratio and phase shift:

$$AR = \frac{K}{\sqrt{1+\tau^2\omega^2}}, \quad \phi = \tan^{-1}(\tau\omega)$$

➡ Same amplitude ratio but positive phase shift

- First order unstable system



- Overall characteristics of Bode plots
  - dc Gain is the value of *AR* at $\omega = 0$
  - Slope of high frequency asymptote is the negative of the relative order of the system
  - Inflection points in *AR* and the phase shift correspond to poles and zeros of the transfer function
  - Each unstable poles and stable zeros lead to positive phase shift of 90 degrees as $\omega \rightarrow \infty$
  - Each stable pole and Unstable zeros reduce phase shift by 90 degrees as $\omega \rightarrow \infty$
  - Delay leads to a phase shift of $-\infty$ as $\omega \rightarrow \infty$