2.9 Linearization

Linearization is the process of finding a linear model that approximates a nonlinear one.

Case I One and Two Variable Versions

1. Taylor series expansion for one variable

\[ f(x) = f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \cdots \text{ Higher Order Terms} \]

Linearized Version: \[ f(x) \approx f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} \]

2. Taylor's Series for Two Variables

\[ f(x,u) = f(x_0,u_0) + f'(x,u)\bigg|_{x_0,u_0} (x-x_0) + f'(x,u)\bigg|_{x_0,u_0} (u-u_0) + \cdots \text{ Higher Order Terms} \]

Linearized Version: \[ f(x,u) \approx f(x_0,u_0) + f'(x,u)\bigg|_{x_0,u_0} (x-x_0) + f'(x,u)\bigg|_{x_0,u_0} (u-u_0) \]

Case II General vector version

Determine equilibrium values of \( x_0, u_0 \), that is, values where \( \dot{x}_0 = 0 = f(x_0,u_0) \). We then expand the nonlinear equations in terms of perturbations from these equilibrium values: that is, we let \( x = x_0 + \delta x \) and \( u = u_0 + \delta u \), so that

\[ \dot{x}_0 + \delta \dot{x} \equiv f(x_0,u_0) + F \delta x + G \delta u \]

where \( F \) and \( G \) are the linearized matrices for the function \( f(x,u) \) at \( x_0 \) and \( u_0 \).
Example
1. Sinusoidal Function
The motion of the simple pendulum system is described by the differential equation
\[(l + R\theta)\ddot{\theta} + g\sin\theta = 0\]
\[l = \text{length of the cord in the vertical (down) position},\]
\[R = \text{radius of the cylinder.}\]
Linearize the equation around the point \(\theta = 0\).

2. Polynomial Function
Find the linearization \(L(x)\) of \(f(x)\) at \(x = 2\).
\[f(x) = x^3 - 2x + 3\]

SOLUTION
Example
1. Sinusoidal Function
The motion of the simple pendulum system is described by the differential equation

\[(l + R \theta)\ddot{\theta} + g \sin \theta = 0\]

\(l = \) length of the cord in the vertical (down) position,
\(R = \) radius of the cylinder.
Linearize the equation around the point \( \theta = 0 \).

**SOLUTION**
Nonlinear term \( \sin \theta \) term should be linearized. By using Equation (1.1), \( \sin \theta \) can be approximated as

\[
\sin \theta = \sin \theta_0 + \cos \theta_0 (\theta - \theta_0) \quad \text{where} \quad \theta_0 = 0
\]

\[
= \sin(0) + \cos(0) (\theta - 0)
\]

\[= \theta\]

For small values of \( \theta \), the system equation reduces to an equation for a simple pendulum,

\[\ddot{\theta} + \frac{g}{l} \theta = 0.\]

2. Polynomial Function
Find the linearization \( L(x) \) of \( f(x) \) at \( x = 2 \).
\[f(x) = x^3 - 2x + 3\]

**SOLUTION**
Since the function \( f(x) \) has only one variable \( x \), we can use

\[f(x) \approx f(x_0) + \frac{f'(x_0)(x - x_0)}{1!}\]

where \( f'(x) = 3x^2 - 2, x_0 = 2 \)

Therefore,

\[f(x) \approx f(2) + \frac{f'(2)(x - 2)}{1!}\]

\[\approx 10x - 13\]
Linearize the equation \[ f(x) = 2x^5 - 2x^2 + 3x + 4 \]
Linearize the equation

\[ \ddot{x} + 3x^2 \dot{x} + x^2 + 3 \cdot \sin(x) = 0 \]
2.10 MODELING – SUMMARY

- For linear time-invariant (LTI) systems the ‘model’ consists of a set of ordinary differential eqns. with constant coefficients.
- Examples:

1). Standard form of first order system

\[ \tau \dot{y} + y = Ku \]

Laplace transforming (with zero ICs)

\[ \tau s Y(s) + Y(s) = Ku(s) \]

\[ \Rightarrow (\tau s + 1)Y(s) = Ku(s) \Rightarrow \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1} \]

\( K = \text{Gain} \)
\( \tau = \text{time constant} \)

2). Standard form of a second order system

\[ \frac{K}{s^2 + 2\zeta \omega_n s + \omega_n^2} \]

\( K = \text{Gain} \)
\( \omega_n = \text{natural freq., rad/s} \)
\( \zeta = \text{damping ratio} \)