LECTURE 6

3.1 TWO TYPICAL INPUT SIGNALS USED IN ANALYSIS

UNIT IMPULSE FUNCTION

Consider the function $f_\Delta(x;a)$ shown in Fig. 1 and defined by Eqns. 1 and 2,

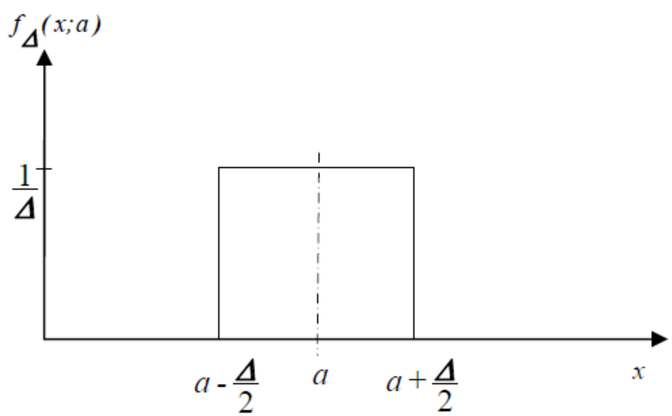
$$f_\Delta(x;a)\begin{cases} 0 & -\infty < x < a - \frac{\Delta}{2} \\ \frac{1}{\Delta} & a - \frac{\Delta}{2} \le x \le a + \frac{\Delta}{2} \\ 0 & a + \frac{\Delta}{2} < x < \infty \end{cases} \qquad (1)$$

The function has the unique property that

$$\int_{-\infty}^{\infty} f_\Delta(x;a)dx = 1 \qquad (2)$$

Taking the limit of $f_\Delta(x;a)$, as $\Delta \to 0$ yields

$$\lim_{\Delta\to 0} f_\Delta(x;a) = \delta(x-a) = \begin{cases} 0 & x \ne a \\ \infty & x = a \end{cases} \qquad (3)$$



From Eq. (2)

$$\int_{-\infty}^{\infty} \delta(x-a)dx = 1 \qquad (4)$$

The function defined by Eq. (3) and (4) is called the unit impulse function (approximate forms are eqns. 1 and 2). It has many applications in physics and engineering. It is used to mathematically represent physical quantities that act for a short time, i.e., an impulsive force, and a concentrated load applied at a location $x = a$ on a structure.

(ADAPTED FROM SATISH NAIR)

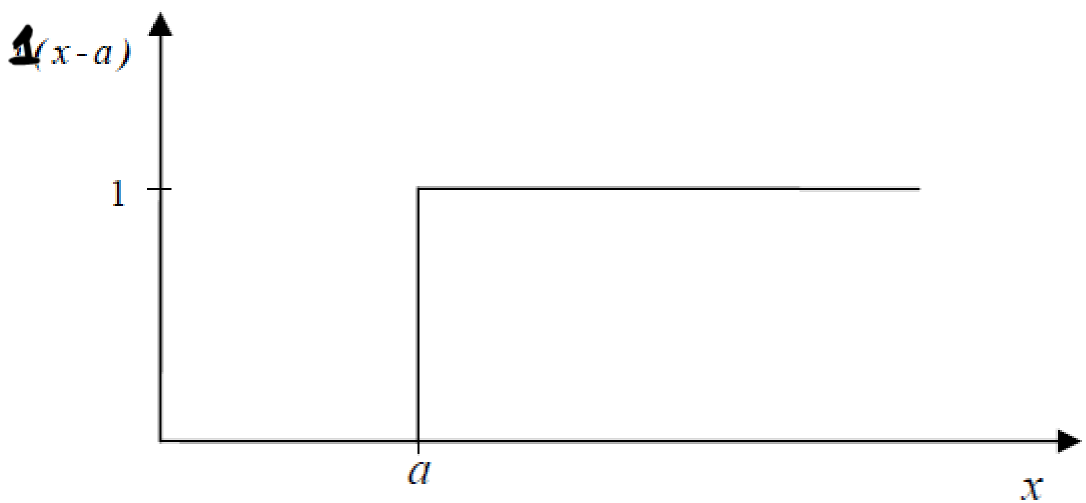
UNIT STEP FUNCTION : $\mathbb{1}(t) = \begin{cases} 0 & t < 0 \\ 1, & t \geq 0 \end{cases}$

Now define

$$\mathbb{1}(x-a) = \int_0^x \delta(x-a)dx = \int_0^x \lim_{\Delta \to 0} f_\Delta(x;a)dx = \begin{cases} 0 & x \leq a \\ 1 & x > a \end{cases} \qquad (5)$$

The function defined in Eq. (5) is called the unit step function and is plotted in Fig. 2. Differentiating Eq. (5) gives

$$\frac{d\mathbb{1}(x-a)}{dx} = \delta(x-a) \qquad (6)$$



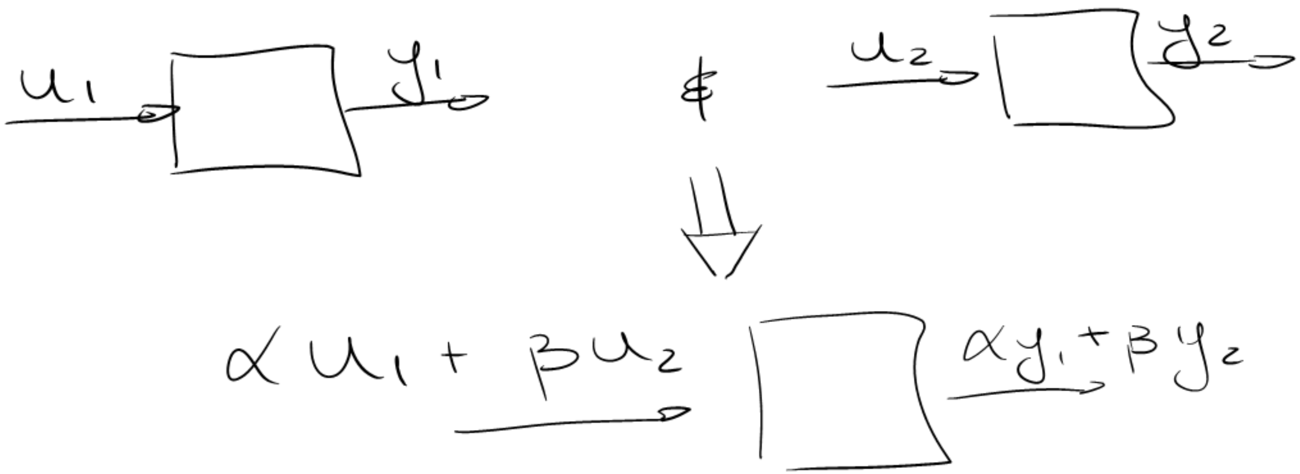
Note: You can integrate a step to get a ramp, and a ramp to get a parabola, …..and so on.

The definitions of the unit impulse function and unit step function can also be used to derive the following integral formulas. For any function ($g(t)$),

$$\int_0^t \delta(\tau - a)g(\tau)d\tau = 1(t-a)g(a) \tag{7}$$

$$\int_0^t 1(\tau - a)g(\tau)d\tau = 1(t-a)\int_a^t g(\tau)d\tau \tag{8}$$

SUPERPOSITION



Ex SHOW THAT SUPERPOSITION HOLDS FOR

$$\dot y + Ky = u$$

$$\dot y_1 + Ky_1 = u_1 \Rightarrow \alpha(\dot y_1 + Ky_1) = \alpha u_1$$

$$\dot y_2 + Ky_2 = u_2 \Rightarrow \beta(y_2 + K_2 y_2) = \beta u_2$$

$$\alpha(\dot y_1 + Ky_1 - u_1) + \beta(\dot y_2 + Ky_2 - u_2) = 0$$

$$0 + 0 = 0 \checkmark$$

TIME INVARIANCE

$$u(t) \longrightarrow \boxed{} \longrightarrow y(t) \quad \Rightarrow \quad u(t-\tau) \longrightarrow \boxed{} \longrightarrow y(t-\tau)$$

Ex: Show that $\dot{y}(t) + K(t)\,y(t) = u(t)$ is TI?

$$\dot{y}_1(t) + K(t)\,y_1(t) = u_1(t) \qquad (1)$$

$$\dot{y}_2(t) + K(t)\,y_2(t) = u_1(t-\tau)$$

if $y_2(t) = y_1(t-\tau) \Rightarrow \dot{y}_1(t-\tau) + K(t)\,y_1(t-\tau) = u_1(t-\tau)$

for $\eta = t-\tau \Rightarrow \dot{y}_1(\eta) + K(\eta+\tau)\,y_1(\eta) = u_1(\eta)$

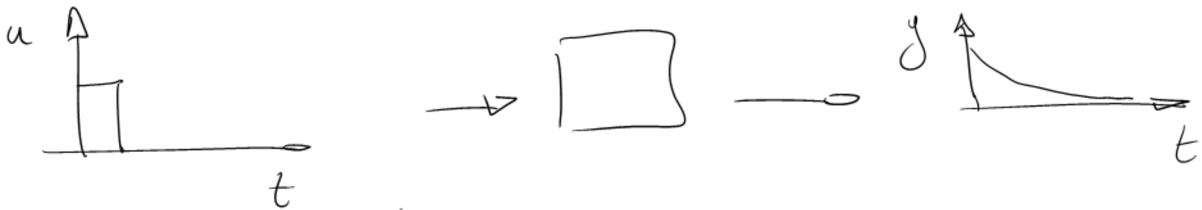
which is $=$ (1) only if $\tau = 0$ or

$K(\eta+\tau) = K(\eta) \quad \forall \tau$

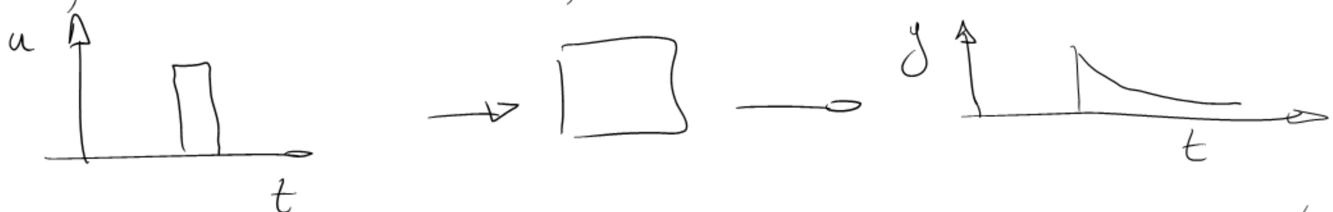
$\Rightarrow K = c^{te}$

IF WE PUT TOGETHER THESE FOUR CONCEPTS (IMPULSE, STEP, SUPERPOSITION & TIME INVARIANCE) . . .

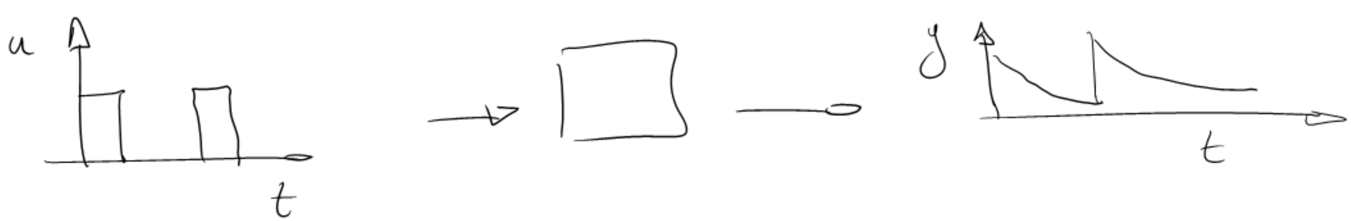
IMAGINE THAT FOR AN INPUT $u(t)$, OUR LTI SYSTEM RESPONDS W/ $y(t)$ AS SO:



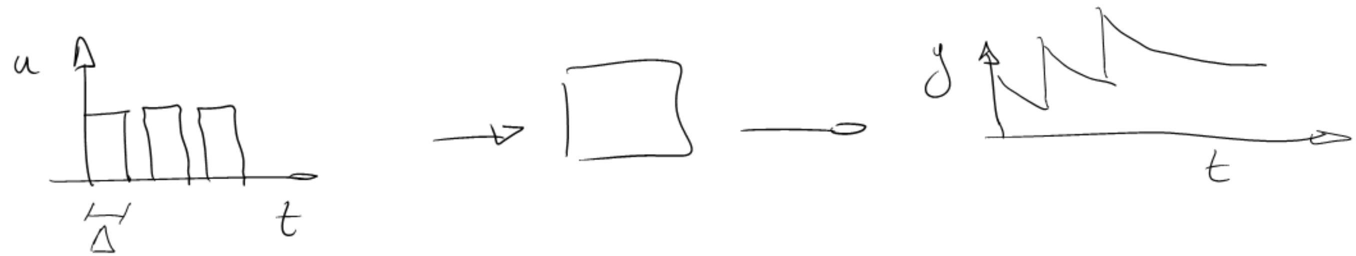
SO, FOR $u(t-\tau)$, THE OUTPUT WOULD BE



BY SUPERPOSITION, IF $u(t)$ IS THE TWO $u(t)$'S ABOVE:



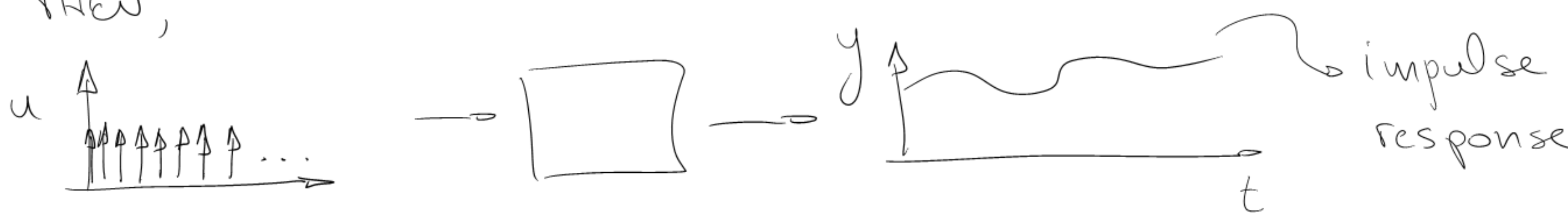
AND GENERALIZING THIS IDEA...



NOW, IF THE WIDTH OF THE PULSES $u(t)$ BECOME VERY NARROW ($\lim_{\Delta \to 0}$ ⇒ THAT IS: $\delta(t)$ = UNIT IMPULSE FUNCTION)

AND THE DISTANCE BTW THEM ALSO VERY SMALL,

THEN,



FOR A CONTINUOUS FUNTION $f(t)$:

$$\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau = f(t)$$

SO, IN THE CASE OF A LTI SYSTEM, WE

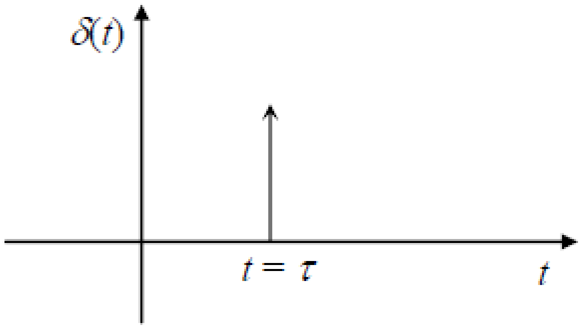
SAY THAT

$$\int_{-\infty}^{\infty} \underbrace{h(\tau)}_{\text{impulse response}}\,\delta(t-\tau)\,\delta\tau = y(t)$$

3.2 SUPERPOSITION AND CONVOLUTION INTERGRALS

- Dirac delta function or unit impulse function (mathematical fiction!)

$$\delta(t-\tau) = \begin{cases} \infty & t = \tau \\ 0 & t \neq \tau \end{cases}$$



and

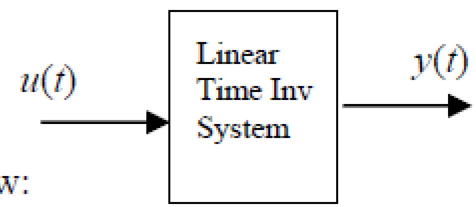
$$\int_{-\infty}^{\infty} \delta(t-\tau)d\tau = 1$$

If $f(t)$ is continuous at $t = \tau$, this implies that

$$\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau = f(t)$$

- Impulse response: Response of a system to a unit impulse input. For a linear time invariant (constant coefficient) system, the impulse response is designated as *h(t)* by convention. So, what is the response at *t* to an impulse applied at τ ? It is $h(t-\tau)$, i.e., it is just time shifted version of *h(t)* (starts at τ instead of 0), as one might expect anyway! Now, the response to any input *u(t)* can be obtained using superposition if *u(t)* is broken up into lots of impulses! How do we do that? Can you imagine the time domain picture of *u(t)* as a series of discrete impulses?

$$y(t) = \lim_{\Delta\tau \to 0} \sum_{\tau=-\infty}^{\infty} [u(\tau)d\tau]h(t-\tau) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

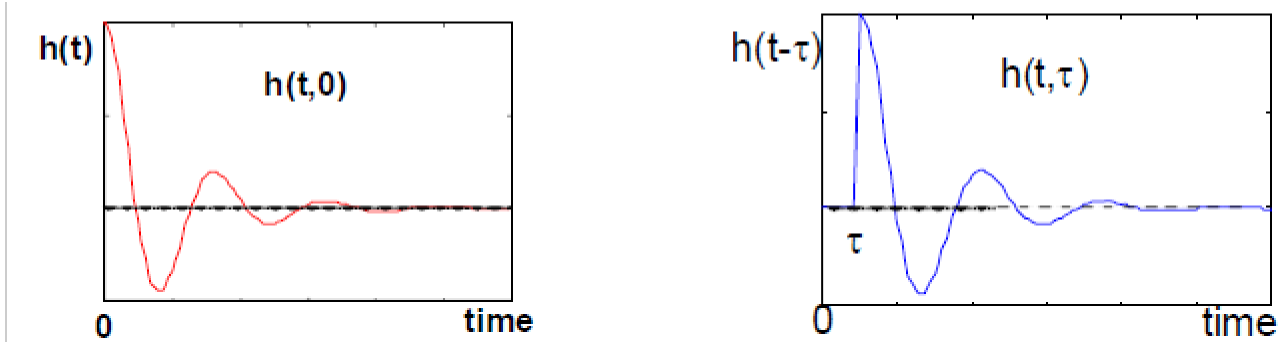


Know:

$u = \delta(t-\tau) \Rightarrow y(t) = h(t-\tau)$

unit impulse at time τ — impulse response of th system, i.e., response a impulse at τ.

This is the **superposition integral**.



The superposition integral provides the response for LTI systems, as

$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau \quad \text{or} \quad y(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

$\eta = t - \tau$ $\qquad$ $\eta = \tau$

This is called the **convolution integral**.

$$y(t) = \int_{\infty}^{-\infty} u(t-\eta)h(\eta)(-d\eta) = \int_{-\infty}^{\infty} u(t-\eta)h(\eta)d\eta$$

The convolution integral occurs so often that the notation $y = u * h$ is often used to represent the equation.

<u>Utility</u>: The response of the system to ANY time varying input can be determined using this convolution integral.

[For a general linear system, the impulse response can be represented as $h(t,\tau)$ and it is not equal to $h(t-\tau)$ …with that change, all the development above holds.]

Ex

$$\dot{y} + Ky = u = \delta(t) \qquad y(0) = 0$$

$$\int_{0^-}^{0^+} (\dot{y} + Ky)\, dt = \int_{0^-}^{0^+} \delta(t)\, dt$$

$$\cancel{\int_{0^-}^{0^+} \dot{y}\, dt}^{\,y} + \cancel{K\int_{0^-}^{0^+} y}^{\,0} = 1$$

$$y\Big|_{0^-}^{0^+} = 1 \Rightarrow y(0^+) - \cancel{y(0^-)}^{\,0} = 1$$

Assume $y = Ae^{st} \Rightarrow \dot{y} = sAe^{st}$

$$Ase^{st} + KAe^{st} = 0 \Rightarrow s + K = 0$$

$$s = -K$$

$$y(0^+) = 1 \quad \& \quad y(t) = A e^{st} \Rightarrow A = 1$$

$$y(t) \triangleq h(t) = e^{-kt} \quad \text{since } h(t) \not\exists \text{ for } t<0$$

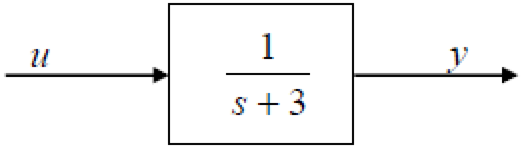
$$h(t) = e^{-kt} \ast 1(t) \qquad 1(t) = \begin{cases} 0 & t<0 \\ 1 & t \geq 0 \end{cases}$$

$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau \quad \text{or} \quad y(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$$

$$y(t) = \int_{-\infty}^{\infty} \delta(t-\tau)\, e^{-k\tau}\, 1(\tau)\, d\tau =$$

$$= \int_{0}^{\infty} \delta(t-\tau)\, e^{-k\tau}\, d\tau$$

Example: Given the following system:



Find the response y if u is a unit step?

Solution: The unit step input u can be expressed as

$$u(\tau) = \begin{cases} 1 & \tau \geq 0 \\ 0 & \tau < 0 \end{cases}$$

The impulse response of this system is given by

$h = e^{-3t}$ (simply the inverse Laplace transform of the transfer function, since L(impulse)=1)

Since the system is linear and time invariant, $h(t,\tau)=h(t-\tau)$.
For this problem, $h(t-\tau)= e^{-3(t-\tau)}$.

Now perform the convolution operation to get the output *y*

$$y(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$$

$$= \int_0^t 1 \cdot e^{-3(t-\tau)} d\tau$$

$$= \frac{1}{3}(1-e^{-3t})$$

Note that when t < τ, $h(t-\tau) = 0$.

MATLAB Commands:

```
syms s t tau
G=1/(s+3);
u=1;
h=ilaplace(G);
h_tau=subs(h,t,t-tau);
y=int(h_tau*u,tau,0,t)
```

Laplace Transform

$$Y(s) = \int_{-\infty}^{\infty} y(t) e^{-st} \, dt$$

but if $y(t) = \int_{-\infty}^{\infty} h(t) u(t-\tau) \, d\tau$

$$Y(s) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} h(t) u(t-\tau) \, d\tau \right] e^{-st} \, dt$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[u(t-\tau) e^{-st} \, dt \right] h(\tau) \, d\tau$$

$$Y(s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[u(\eta)\, e^{-s(\eta+\tau)}\, d\eta \right] h(\tau)\, d\tau$$

where $\eta = t - \tau$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(\eta)\, e^{-s\eta}\, d\eta \;\; e^{-s\tau} h(\tau)\, d\tau$$

$$= \underbrace{\int_{-\infty}^{\infty} u(\eta)\, e^{-s\eta}\, d\eta}_{U(s)} \;\; \underbrace{\int_{-\infty}^{\infty} e^{-s\tau} h(\tau)\, d\tau}_{H(s)}$$

$$Y(s) = U(s) * H(s)$$