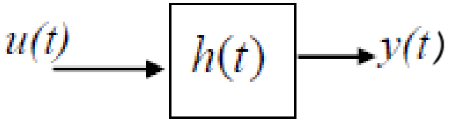
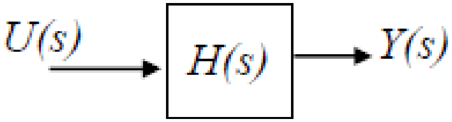
LECTURE 7

3.3 TRANSFER FUNCTIONS





- Using the convolution integral, let $u(t) = e^{st}$

$$\Rightarrow y(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st} * \left[\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\right]$$

$$\Rightarrow y(t) = e^{st}H(s) \qquad \text{where } H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

So, the transfer function of a system = Laplace transform of the impulse response of the system.

For a causal system, $h(t) = 0$ for $t < 0$, so, we can write

$$H(s) = \int_0^{\infty} h(\tau) e^{-st} d\tau$$

(ADAPTED FROM SATISH NAIR)

ALSO, $y(t) = \int_0^\infty h(\tau)\, u(t-\tau)\, d\tau$

Ex COMPUTE THE TRANSFER FUNCTION FOR $\dot{y} + Ky = u(t)$ for $u(t) = e^{st}$

THAT IS: $\dot{y} + Ky = e^{st}$

by definition

$\Rightarrow$ $y(t) = \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau =$

$= \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \cdot e^{st}\, d\tau = e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau}_{H(s)}$

So, $y(t) = e^{st} H(s)$

THEN, $\dot{y}(t) = s e^{st} H(s)$

$\underbrace{s\cancel{e^{st}} H(s)}_{\dot{y}} + \underbrace{K\cancel{e^{st}} H(s)}_{Ky} = \underbrace{\cancel{e^{st}}}_{u}$

SOLVING FOR $H(s)$

$\Rightarrow$ $\boxed{H(s) = \frac{1}{s+K}}$ & $\boxed{y(t) = \frac{e^{st}}{s+K}}$

EXPANDING THIS IDEA, IF $u(t) = e^{st}$ AND

$$\dddot{y} + a_1 \ddot{y} + a_2 \dot{y} + a_3 y = b_1 \ddot{u} + b_2 \dot{u} + b_3 u$$

OR $s^3 \cancel{e^{st}} H(s) + a_1 s^2 \cancel{e^{st}} H(s) + a_2 s \cancel{e^{st}} H(s) + a_3 \cancel{e^{st}} =$

$$b_1 s^2 \cancel{e^{st}} + b_2 s \cancel{e^{st}} + b_3 \cancel{e^{st}}$$

$$\Rightarrow \quad H(s) = \frac{Y(s)}{U(s)} = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

IN GENERAL, FOR $\overset{(\circ)^n}{y} + a_1 \overset{(\circ)^{n-1}}{y} + a_2 \overset{(\circ)^{n-2}}{y} \cdots = b_1 \overset{(\circ)^{n-1}}{u} + b_2 \overset{(\circ)^{n-2}}{u} + \cdots$

$$H(s) = \frac{b_1 s^{n-1} + b_2 s^{n-2} + b_3 s^{n-3} \cdots}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} \cdots}$$

NOTE: USING LT

$$s^n Y(s) + a_1 s^{n-1} Y(s) + a_2 s^{n-2} Y(s) \ldots = s^{n-1} U(s) + s^{n-2} U(s) + s^{n-3} U(s) \ldots$$

- FREQUENCY RESPONSE: Find the response to a sinusoid by this method. Note that $A\cos\omega t = \frac{A}{2}(e^{j\omega t} + e^{-j\omega t})$. Let $s = j\omega$ in the formula above to get the response to $u(t) = e^{j\omega t}$ as $y(t) = H(j\omega)e^{j\omega t}$; and, for $u(t) = e^{-j\omega t}$ the response is $y(t) = H(-j\omega)e^{-j\omega t}$. So, we can get the response $y(t)$ to $u(t) = A\cos\omega t = \frac{A}{2}(e^{j\omega t} + e^{-j\omega t})$, by superposition as

$$y(t) = \frac{A}{2}\left[H(j\omega)e^{j\omega t} + H(-j\omega)e^{-j\omega t}\right]$$

NOTE:
$M(\omega) = |H(s)|$
$\varphi(\omega) = \angle H(s)$

Writing $H(j\omega) = M(\omega)e^{j\phi(\omega)}$ (Magnitude-phase form), we get

$$y(t) = \frac{A}{2}M(\omega)\{e^{j[\omega t+\phi(\omega)]} + e^{-j[\omega t+\phi(\omega)]}\} = A\cdot M(\omega)\cos(\omega t + \phi(\omega))$$

$\Rightarrow$ If a system represented by the transfer function $H(s)$ has a sinusoidal input with magnitude $A(\omega)$, the output will be sinusoidal at the same frequency with magnitude $A{\cdot}M(\omega)$ and shifted in phase by the angle $\phi(\omega)$, i.e., $\phi_{output}(\omega) = \phi_{input}(\omega) + \phi(\omega)$.

<u>Frequency response curves</u>: M(ω) v/s ω and f(ω) v/s ω

- Consider $y(t) = \int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau$. Taking Laplace transform of both sides yields

$$Y(s) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} u(t-\tau)h(\tau)d\tau\right]e^{-st}dt$$

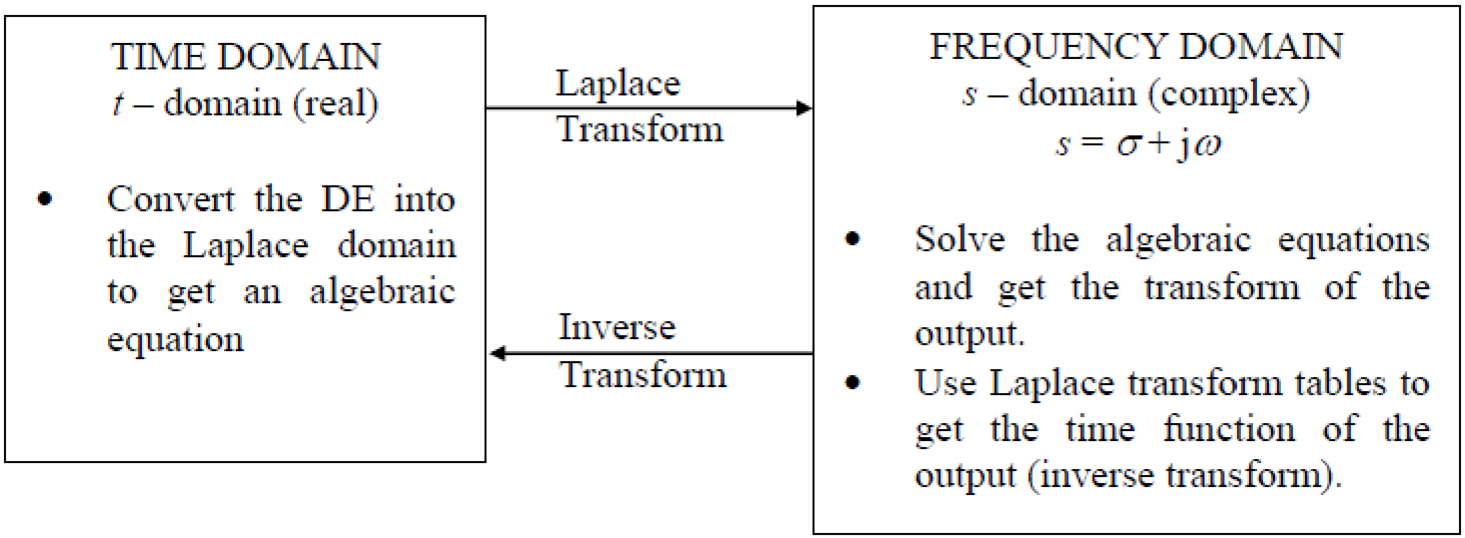
Interchanging the integrals

$$Y(s) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} u(t-\tau)h(\tau)e^{-st}dt\right]d\tau = U(s)\int_{-\infty}^{\infty} e^{-s\tau}h(\tau)d\tau = U(s)H(s)$$

$$\Rightarrow Y(s) = H(s)U(s)$$

3.4 LAPLACE TRANSFORMS

Objective: Solve DEs



- Consider a signal $f(t)$. Its one-side (unilateral) Laplace transform is defined as

$$F(s) = L[f(t)] = \int_{0^-}^{\infty} f(t)e^{-st}dt \qquad \text{where } s = \sigma + j\omega .$$

- Given a DE (or set of DEs), convert it to the Laplace domain. It becomes an algebraic equation now (i.e., no derivative terms exist). Now, by algebraic manipulation, find the transform of the output. Then, using Laplace tables, determine the time function (inverse transform) of the output $\Rightarrow$ SOLVE THE DES USING THE LAPLACE TRANSFORM METHOD.

Problem: Find the Laplace transforms of the step $(a1(t))$, ramp $(bt1(t))$ functions, and the sinusoid $\sin(\omega t)$.

Solution: (i) $f(t) = a1(t)$

$$F(s) = \int_0^\infty ae^{-st}dt = -\frac{ae^{-st}}{s}\bigg|_0^\infty = 0-(-\frac{a}{s}) = \frac{a}{s}$$

(ii) $f(t) = bt1(t)$

$$F(s) = \int_0^\infty bte^{-st}dt = \left(-\frac{bte^{-st}}{s} - \frac{be^{-st}}{s}\right)\bigg|_0^\infty = \frac{b}{s^2}$$

$\left[\int udv = uv - \int vdu\right]$

(iii) $f(t) = \sin\omega t$

$$F(s) = \int_0^\infty \sin\omega t e^{-st}dt = \int_0^\infty \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)e^{-st}dt = \frac{\omega}{s^2+\omega^2}$$

Example: Solve $\ddot{c} + 2\dot{c} + 10c = 10r$ with $r = 1.0$, $c(0) = \dot{c}(0) = 0$.

Step 1: Taking the Laplace transform, we get

$$s^2 C(s) + 2sC(s) + 10C(s) = 10R(s)$$

Step 2: Solving for $C(s)$ yields

$$C(s) = \frac{10R(s)}{s^2 + 2s + 10} = \frac{10}{s^2 + 2s + 10} x \frac{1}{s}$$

since $R = 1/s$.

Steps

- (i) take Laplace transform of both sides (taking care to include the initial conditions) to go to the Laplace domain
- (ii) Solve for the Laplace of the output by algebraic manipulation
- (iii) Take inverse transform – to come back to the time domain

Step 3: In order to look up in the Laplace transform table, the above equation is rewritten as

$$C(s) = \frac{10}{s[(s+1)^2 + 9]}$$

Step 4: Therefore, the time function is given by

$$c(t) = 1 - e^{-t}(\cos 3t + \frac{1}{3}\sin 3t)$$

MATLAB Commands

```
syms s t
C=10/(s^3+2*s^2+10*s);
c=ilaplace(C,s,t)
```

3.5 PROPERTIES OF LAPLACE TRANSFORMS

(i) Superposition: $\mathcal{L}[\alpha f_1(t) + \beta f_2(t)] = \alpha F_1(s) + \beta F_2(s)$

(ii) Time delay: $\mathcal{L}[f(t-\lambda)] = e^{-s\lambda} F(s)$

$f(t-\lambda)$ is the function $f(t)$ delayed by λ time units.

(iii) Time scaling: $\mathcal{L}[f(at)] = \dfrac{1}{|a|} F\left(\dfrac{s}{a}\right)$

(iv) Shift in frequency: $\mathcal{L}[e^{-at} f(t)] = F(s+a)$

(v) Differentiation: $\mathcal{L}[f^m(t)] = s^m F(s) - s^{m-1} f(0^-) - s^{m-2} \dot{f}(0^-) - \cdots - f^{(m-1)}(0^-)$

where f^m denotes the m$^{\text{th}}$ derivative of *f* with respect to time.

(vi) Integration: $\mathcal{L}[\int_0^t f(\xi) d\xi] = \int_0^\infty \left[\int_0^t f(\xi) d\xi \right] e^{-st} dt = \dfrac{F(s)}{s}$

(vii) Convolution: $\mathcal{L}[f_1(t) * f_2(t)] = F_1(s) F_2(s)$

(viii) Time product: $\mathcal{L}[f_1(t) f_2(t)] = F_1(s) * F_2(s)$

(ix) Multiplication by time: $\mathcal{L}[tf(t)] = -\dfrac{d}{ds}(F_2(s))$