\[ x_3 = \frac{1}{|A|} \begin{vmatrix} 2 & 1 & 3 \\ 1 & 3 & 7 \\ 1 & 1 & 1 \end{vmatrix} = \frac{-8}{4} = -2 \]

Example CB.5
Using a Computer, solve Example B.7.
\[ A = \begin{bmatrix} 2 & 1 & 1 \\ 3 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}; \quad b = [3 \ 7 \ 1]^T; \]
for \( k=1:3 \)
\[ A_1 = A; \]
\[ A_1(:,k) = b; \]
\[ D = A_1; \]
\[ x(k) = \text{det}(D)/\text{det}(A); \]
end
\[ x = x'; \]
\[ x = [2 \ 1 \ -2] \]

B.5 Partial Fraction Expansion

In the analysis of linear time-invariant systems, we encounter functions that are ratios of two polynomials in a certain variable, say \( x \). Such functions are known as rational functions. A rational function \( F(x) \) can be expressed as

\[ F(x) = \frac{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1} x^{n-1} + \cdots - a_1 x + a_0} \]  \hspace{1cm} (B.32)

\[ = \frac{P(x)}{Q(x)} \]  \hspace{1cm} (B.33)

The function \( F(x) \) is improper if \( m \geq n \) and proper if \( m < n \). An improper function can always be separated into the sum of a polynomial in \( x \) and a proper function. Consider, for example, the function

\[ F(x) = \frac{2x^3 + 9x^2 + 11x + 2}{x^2 + 4x + 3} \]  \hspace{1cm} (B.34a)

Because this is an improper function, we divide the numerator by the denominator until the remainder has a lower degree than the denominator.

\[
\begin{align*}
2x + 1 \\
x^2 + 4x + 3) 2x^3 + 9x^2 + 11x + 2 \\
2x^3 + 8x^2 + 6x \\
x^2 + 5x + 2 \\
x^2 + 4x + 3 \\
x - 1
\end{align*}
\]
Therefore, $F(x)$ can be expressed as

$$F(x) = \frac{2x^3 + 9x^2 + 11x + 2}{x^2 + 4x + 3} = \frac{2x + 1}{\text{polynomial in } x} + \frac{x - 1}{\text{proper function}}$$

(B.34b)

A proper function can be further expanded into partial fractions. The remaining discussion in this section is concerned with various ways of doing this.

### B.5.1 Partial Fraction Expansion: Method of Clearing Fractions

This method consists of writing a rational function as a sum of appropriate partial fractions with unknown coefficients, which are determined by clearing fractions and equating the coefficients of similar powers on the two sides. This procedure is demonstrated by the following example.

**Example B.8**

Expand the following rational function $F(x)$ into partial fractions:

$$F(x) = \frac{x^3 + 3x^2 + 4x + 6}{(x + 1)(x + 2)(x + 3)^2}$$

This function can be expressed as a sum of partial fractions with denominators $(x + 1)$, $(x + 2)$, $(x + 3)$, and $(x + 3)^2$, as shown below.

$$F(x) = \frac{x^3 + 3x^2 + 4x + 6}{(x + 1)(x + 2)(x + 3)^2} = \frac{k_1}{x + 1} + \frac{k_2}{x + 2} + \frac{k_3}{x + 3} + \frac{k_4}{(x + 3)^2}$$

To determine the unknowns $k_1$, $k_2$, $k_3$, and $k_4$ we clear fractions by multiplying both sides by $(x + 1)(x + 2)(x + 3)^2$ to obtain

\[
x^3 + 3x^2 + 4x + 6 = k_1(x^3 + 8x^2 + 21x + 18) + k_2(x^3 + 7x^2 + 15x + 9) + k_3(x^3 + 6x^2 + 11x + 6) + k_4(x^2 + 3x + 2)
\]

\[
= x^3(k_1 + k_2 + k_3) + x^2(8k_1 + 7k_2 + 6k_3 + k_4) + x(21k_1 + 15k_2 + 11k_3 + 3k_4) + (18k_1 + 9k_2 + 6k_3 + 2k_4)
\]

Equating coefficients of similar powers on both sides yields

$$k_1 + k_2 + k_3 = 1$$
$$8k_1 + 7k_2 + 6k_3 + k_4 = 3$$
$$21k_1 + 15k_2 + 11k_3 + 3k_4 = 4$$
$$18k_1 + 9k_2 + 6k_3 + 2k_4 = 6$$

Solution of these four simultaneous equations yields

$$k_1 = 1, \quad k_2 = -2, \quad k_3 = 2, \quad k_4 = -3$$

Therefore,

$$F(x) = \frac{1}{x + 1} - \frac{2}{x + 2} + \frac{2}{x + 3} - \frac{3}{(x + 3)^2}$$

Although this method is straightforward and applicable to all situations, it is not necessarily the most efficient. We now discuss other methods which can reduce numerical work considerably.
B.5-2 Partial Fractions: The Heaviside "Cover-Up" Method

1. Unrepeated Factors of \( Q(x) \)

We shall first consider the partial fraction expansion of \( F(x) = P(x) / Q(x) \), in which all the factors of \( Q(x) \) are unrepeated. Consider the proper function

\[
F(x) = \frac{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0} \quad m < n
\]

\[
F(x) = \frac{P(x)}{(x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)}
\]

We can show that \( F(x) \) in Eq. (B.35a) can be expressed as the sum of partial fractions

\[
F(x) = \frac{k_1}{x - \lambda_1} + \frac{k_2}{x - \lambda_2} + \cdots + \frac{k_n}{x - \lambda_n}
\]

(B.35b)

To determine the coefficient \( k_1 \), we multiply both sides of Eq. (B.35b) by \( x - \lambda_1 \) and then let \( x = \lambda_1 \). This yields

\[
(x - \lambda_1) F(x) \big|_{x=\lambda_1} = k_1 + \frac{k_2 (x - \lambda_1)}{(x - \lambda_2)} + \frac{k_3 (x - \lambda_1)}{(x - \lambda_3)} + \cdots + \frac{k_n (x - \lambda_1)}{(x - \lambda_n)} \bigg|_{x=\lambda_1}
\]

On the right-hand side, all the terms except \( k_1 \) vanish. Therefore,

\[
k_1 = (x - \lambda_1) F(x) \big|_{x=\lambda_1}
\]

(B.36)

Similarly, we can show that

\[
k_r = (x - \lambda_r) F(x) \big|_{x=\lambda_r} \quad r = 1, 2, \ldots, n
\]

(B.37)

Example B.9

Expand the following rational function \( F(x) \) into partial fractions:

\[
F(x) = \frac{2x^2 + 9x - 11}{(x + 1)(x - 2)(x + 3)} = \frac{k_1}{x + 1} + \frac{k_2}{x - 2} + \frac{k_3}{x + 3}
\]

To determine \( k_1 \), we let \( x = -1 \) in \( (x + 1) F(x) \). Note that \( (x + 1) F(x) \) is obtained from \( F(x) \) by omitting the term \( x + 1 \) from its denominator. Therefore, to compute \( k_1 \) corresponding to the factor \( x + 1 \), we cover up the term \( (x + 1) \) in the denominator of \( F(x) \) and then substitute \( x = -1 \) in the remaining expression. (Mentally conceal the term \( x + 1 \) in \( F(x) \) with a finger and then let \( x = -1 \) in the remaining expression.) The procedure is explained step by step below.

\[
F(x) = \frac{2x^2 + 9x - 11}{(x + 1)(x - 2)(x + 3)}
\]

Step 1: Cover up (conceal) the factor \( x + 1 \) from \( F(x) \):
\[
\frac{2x^2 + 9x - 11}{(x + 1)(x - 2)(x + 3)}
\]

Step 2: Substitute \( x = -1 \) in the remaining expression to obtain \( k_1 \):

\[
k_1 = \frac{2 - 9 - 11}{(-1 - 2)(-1 + 3)} = \frac{-18}{-6} = 3
\]

Similarly, to compute \( k_2 \), we cover up the factor \((x - 2)\) in \( F(x) \) and let \( x = 2 \) in the remaining function, as shown below.

\[
k_2 = \frac{2x^2 + 9x - 11}{(x + 1)(x - 2)(x + 3)} \bigg|_{x=2} = \frac{8 + 18 - 11}{(2 + 1)(2 + 3)} = \frac{15}{15} = 1
\]

And

\[
k_3 = \frac{2x^2 + 9x - 11}{(x + 1)(x - 2)(x + 3)} \bigg|_{x=-3} = \frac{18 - 27 - 11}{(-3 + 1)(-3 - 2)} = \frac{-20}{10} = -2
\]

Therefore,

\[
F(x) = \frac{2x^2 + 9x - 11}{(x + 1)(x - 2)(x + 3)} = \frac{3}{x + 1} + \frac{1}{x - 2} - \frac{2}{x + 3}
\]

**Complex Factors in \( F(x) \)**

The procedure above works regardless of whether the factors of \( Q(x) \) are real or complex. Consider, for example,

\[
F(x) = \frac{4x^2 + 2x + 18}{(x + 1)(x^2 + 4x + 13)}
\]

\[
= \frac{4x^2 + 2x + 18}{(x + 1)(x + 2 - j3)(x + 2 + j3)}
\]

\[
= \frac{k_1}{x + 1} + \frac{k_2}{x + 2 - j3} + \frac{k_3}{x + 2 + j3}
\]

where

\[
k_1 = \left[ \frac{4x^2 + 2x + 18}{(x + 1)(x^2 + 4x + 13)} \right]_{x=-1} = 2
\]

Similarly,

\[
k_2 = \left[ \frac{4x^2 + 2x + 18}{(x + 1)(x + 2 - j3)(x + 2 + j3)} \right]_{x=-2+j3} = 1 + j2 = \sqrt{5}e^{j63.43^\circ}
\]

\[
k_3 = \left[ \frac{4x^2 + 2x + 18}{(x + 1)(x + 2 - j3)(x + 2 + j3)} \right]_{x=-2-j3} = 1 - j2 = \sqrt{5}e^{-j63.43^\circ}
\]

Therefore,

\[
F(x) = \frac{2}{x + 1} + \frac{\sqrt{5}e^{j63.43^\circ}}{x + 2 - j3} + \frac{\sqrt{5}e^{-j63.43^\circ}}{x + 2 + j3}
\]

(B.39)
The coefficients $k_2$ and $k_3$ corresponding to the complex conjugate factors are also conjugates of each other. This is generally true when the coefficients of a rational function are real. In such a case, we need to compute only one of the coefficients.

2. Quadratic Factors

Often we are required to combine the two terms arising from complex conjugate factors into one quadratic factor. For example, $F(x)$ in Eq. (B.38) can be expressed as

$$F(x) = \frac{4x^2 + 2x + 18}{(x + 1)(x^2 + 4x + 13)} = \frac{k_1}{x + 1} + \frac{c_1 x + c_2}{x^2 + 4x + 13}$$

The coefficient $k_1$ is found by the Heaviside method to be 2. Therefore,

$$\frac{4x^2 + 2x + 18}{(x + 1)(x^2 + 4x + 13)} = \frac{2}{x + 1} + \frac{c_1 x + c_2}{x^2 + 4x + 13}$$  \hspace{1cm} (B.40)

The values of $c_1$ and $c_2$ are determined by clearing fractions and equating the coefficients of similar powers of $x$ on both sides of the resulting equation. Clearing fractions on both sides of Eq. (B.40) yields

$$4x^2 + 2x + 18 = 2(x^2 + 4x + 13) + (c_1 x + c_2)(x + 1)$$

$$= (2 + c_1)x^2 + (8 + c_1 + c_2)x + (26 + c_2)$$  \hspace{1cm} (B.41)

Equating terms of similar powers yields $c_1 = 2, c_2 = -8$, and

$$\frac{4x^2 + 2x + 18}{(x + 1)(x^2 + 4x + 13)} = \frac{2}{x + 1} + \frac{2x - 8}{x^2 + 4x + 13}$$  \hspace{1cm} (B.42)

Short-Cuts

The values of $c_1$ and $c_2$ in Eq. (B.40) can also be determined by using short-cuts. After computing $k_1 = 2$ by the Heaviside method as before, we let $x = 0$ on both sides of Eq. (B.40) to eliminate $c_1$. This gives us

$$\frac{18}{13} = 2 + \frac{c_2}{13}$$

Therefore,

$$c_2 = -8$$

To determine $c_1$, we multiply both sides of Eq. (B.40) by $x$ and then let $x \to \infty$. Remember that when $x \to \infty$, only the terms of the highest power are significant. Therefore,

$$4 = k_1 + c_1 = 2 + c_1$$

and

$$c_1 = 2$$

In the procedure discussed here, we let $x \to 0$ to determine $c_2$ and then multiply both sides by $x$ and let $x \to \infty$ to determine $c_1$. However, nothing is sacred about these values ($x \to 0$ or $x \to \infty$). We use them because they reduce the number of
computations involved. We could just as well use other convenient values for \(x\), such as \(x = 1\). Consider the case

\[
F(x) = \frac{2x^2 + 4x + 5}{x(x^2 + 2x + 5)} = \frac{k}{x} + \frac{c_1 x + c_2}{x^2 + 2x + 5}
\]

We find \(k = 1\) by the Heaviside method in the usual manner. As a result,

\[
\frac{2x^2 + 4x + 5}{x(x^2 + 2x + 5)} = \frac{1}{x} + \frac{c_1 x + c_2}{x^2 + 2x + 5} \tag{B.43}
\]

To determine \(c_1\) and \(c_2\), if we try letting \(x = 0\) in Eq. (B.43), we obtain \(\infty\) on both sides. So let us choose \(x \approx 1\). This yields

\[
F(1) = \frac{11}{8} = 1 + \frac{c_1 + c_2}{8}
\]

or

\[
c_1 + c_2 = 3
\]

We can now choose some other value for \(x\), such as \(x = 2\), to obtain one more relationship to use in determining \(c_1\) and \(c_2\). In this case, however, a simple method is to multiply both sides of Eq. (B.43) by \(x\) and then let \(x \to \infty\). This yields

\[
2 = 1 + c_1
\]

so that

\[
c_1 = 1 \quad \text{and} \quad c_2 = 2
\]

Therefore,

\[
F(x) = \frac{1}{x} + \frac{x + 2}{x^2 + 2x + 5}
\]

### B.5-3 Repeated Factors in \(Q(x)\)

If a function \(F(x)\) has a repeated factor in its denominator, it has the form

\[
F(x) = \frac{P(x)}{(x - \lambda)^r(x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_j)} \tag{B.44}
\]

Its partial fraction expansion is given by

\[
F(x) = \frac{a_0}{(x - \lambda)^r} + \frac{a_1}{(x - \lambda)^{r-1}} + \cdots + \frac{a_{r-1}}{(x - \lambda)}
\]

\[
+ \frac{k_1}{x - \alpha_1} + \frac{k_2}{x - \alpha_2} + \cdots + \frac{k_j}{x - \alpha_j} \tag{B.45}
\]

The coefficients \(k_1, k_2, \ldots, k_j\) corresponding to the unrepeated factors in this equation are determined by the Heaviside method, as before [Eq. (B.37)]. To find the
coefficients \( a_0, a_1, a_2, \ldots, a_{r-1} \), we multiply both sides of Eq. (B.45) by \((x - \lambda)^r\). This gives us

\[
(x - \lambda)^r F(x) = a_0 + a_1 (x - \lambda) + a_2 (x - \lambda)^2 + \cdots + a_{r-1} (x - \lambda)^{r-1} + k_1 \frac{(x - \lambda)^r}{x - \alpha_1} + k_2 \frac{(x - \lambda)^r}{x - \alpha_2} + \cdots + k_n \frac{(x - \lambda)^r}{x - \alpha_n}
\] (B.46)

If we let \( x = \lambda \) on both sides of Eq. (B.46), we obtain

\[
(x - \lambda)^r F(x) \bigg|_{x = \lambda} = a_0
\] (B.47a)

Therefore, \( a_0 \) is obtained by concealing the factor \((x - \lambda)^r\) in \( F(x) \) and letting \( x = \lambda \) in the remaining expression (the Heaviside “cover up” method). If we take the derivative (with respect to \( x \)) of both sides of Eq. (B.46), the right-hand side is \( a_1 + \) terms containing a factor \((x - \lambda)\) in their numerators. Letting \( x = \lambda \) on both sides of this equation, we obtain

\[
\frac{d}{dx} \left[(x - \lambda)^r F(x)\right] \bigg|_{x = \lambda} = a_1
\]

Thus, \( a_1 \) is obtained by concealing the factor \((x - \lambda)^r\) in \( F(x) \), taking the derivative of the remaining expression, and then letting \( x = \lambda \). Continuing in this manner, we find

\[
a_j = \frac{1}{j!} \frac{d^j}{dx^j} \left[(x - \lambda)^r F(x)\right] \bigg|_{x = \lambda}
\] (B.47b)

Observe that \((x - \lambda)^r F(x)\) is obtained from \( F(x) \) by omitting the factor \((x - \lambda)^r\) from its denominator. Therefore, the coefficient \( a_j \) is obtained by concealing the factor \((x - \lambda)^r\) in \( F(x) \), taking the \( j \)th derivative of the remaining expression, and then letting \( x = \lambda \) (while dividing by \( j! \)).

**Example B.10**

Expand \( F(x) \) into partial fractions if

\[
F(x) = \frac{4x^3 + 16x^2 + 23x + 13}{(x + 1)^3(x + 2)}
\]

The partial fractions are

\[
F(x) = \frac{a_0}{(x + 1)^3} + \frac{a_1}{(x + 1)^2} + \frac{a_2}{x + 1} + \frac{k}{x + 2}
\]

The coefficient \( k \) is obtained by concealing the factor \((x + 2)\) in \( F(x) \) and then substituting \( x = -2 \) in the remaining expression:

\[
k = \frac{4x^3 + 16x^2 + 23x + 13}{(x + 1)^3(x + 2)} \bigg|_{x = -2} = 1
\]

To find \( a_0 \), we conceal the factor \((x + 1)^3\) in \( F(x) \) and let \( x = -1 \) in the remaining expression:

\[
a_0 = \frac{4x^3 + 16x^2 + 23x + 13}{(x + 1)^3(x + 2)} \bigg|_{x = -1} = 2
\]
To find $a_1$, we evaluate the factor $(x+1)^3$ in $F(x)$, take the derivative of the remaining expression, and then let $x = -1$:

$$a_1 = \frac{d}{dx} \left[ \frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)} \right]_{x=-1} = 1$$

Similarly, 

$$a_2 = \frac{1}{2!} \frac{d^2}{dx^2} \left[ \frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)} \right]_{x=-1} = 3$$

Therefore, 

$$F(x) = \frac{2}{(x+1)^3} + \frac{1}{(x+1)^2} + \frac{3}{x+1} + \frac{1}{x+2} \quad \blacksquare$$

### B.5-4 A Hybrid Method: Mixture of the Heaviside "Cover-Up" and Clearing Fractions

For multiple roots, especially of higher order, the Heaviside expansion method, which requires repeated differentiation, can become cumbersome. For a function which contains several repeated and unRepeated roots, a hybrid of the two procedures proves the best. The simpler coefficients are determined by the Heaviside method, and the remaining coefficients are found by clearing fractions or short-cuts, thus incorporating the best of the two methods. We demonstrate this procedure by solving Example B.10 once again by this method.

In Example B.10, coefficients $k$ and $a_0$ are relatively simple to determine by the Heaviside expansion method. These values were found to be $k_1 = 1$ and $a_0 = 2$. Therefore,

$$\frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)} = \frac{2}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{a_2}{x+1} + \frac{1}{x+2}$$

We now multiply both sides of the above equation by $(x+1)^3(x+2)$ to clear the fractions. This yields

$$4x^3 + 16x^2 + 23x + 13 = 2(x+2) + a_1(x+1)(x+2) + a_2(x+1)^2(x+2) + (x+1)^2$$

$$= (1 + a_2)x^3 + (a_1 + 4a_2 + 3)x^2 + (5 + 3a_1 + 5a_2)x + (4 + 2a_1 + 2a_2 + 1)$$

Equating coefficients of the third and second powers of $x$ on both sides, we obtain

1. $1 + a_2 = 4 \quad a_2 = 3$
2. $a_1 + 4a_2 + 3 = 16 \quad a_1 = 1$

We may stop here if we wish because the two desired coefficients, $a_1$ and $a_2$, are now determined. However, equating the coefficients of the two remaining powers of $x$ yields a convenient check on the answer. Equating the coefficients of the $x^1$ and $x^0$ terms, we obtain

$$23 = 5 + 3a_1 + 5a_2$$
$$13 = 4 + 2a_1 + 2a_2 + 1$$
These equations are satisfied by the values \( a_1 = 1 \) and \( a_2 = 3 \), found earlier, providing an additional check for our answers. Therefore,

\[
F(x) = \frac{2}{(x+1)^3} + \frac{1}{(x+1)^2} + \frac{3}{x+1} + \frac{1}{x+2}
\]

which agrees with the previous result.

**A Mixture of the Heaviside "Cover-Up" and Short Cuts**

In the above example, after determining the coefficients \( a_0 = 2 \) and \( k = 1 \) by the Heaviside method as before, we have

\[
\frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)} = \frac{2}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{a_2}{x+1} + \frac{1}{x+2}
\]

There are only two unknown coefficients, \( a_1 \) and \( a_2 \). If we multiply both sides of the above equation by \( x \) and then let \( x \to \infty \), we can eliminate \( a_1 \). This yields

\[
4 = a_2 + 1 \quad \Rightarrow \quad a_2 = 3
\]

Therefore,

\[
\frac{4x^3 + 16x^2 + 23x + 13}{(x+1)^3(x+2)} = \frac{2}{(x+1)^3} + \frac{a_1}{(x+1)^2} + \frac{3}{x+1} + \frac{1}{x+2}
\]

There is now only one unknown \( a_1 \), which can be readily found by setting \( x \) equal to any convenient value, say \( x = 0 \). This yields

\[
\frac{13}{2} = 2 + a_1 + 3 + \frac{1}{2} \quad \Rightarrow \quad a_1 = 1
\]

which agrees with our earlier answer.

**B.5-5 Improper \( F(x) \) with \( m = n \)**

A general method of handling an improper function is indicated in the beginning of this section. However, for a special case where the numerator and denominator polynomials of \( F(x) \) are of the same degree (\( m = n \)), the procedure is the same as that for a proper function. We can show that for

\[
F(x) = \frac{b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0}{x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0}
\]

\[
= b_n + \frac{k_1}{x - \lambda_1} + \frac{k_2}{x - \lambda_2} + \cdots + \frac{k_n}{x - \lambda_n}
\]

the coefficients \( k_1, k_2, \ldots, k_n \) are computed as if \( F(x) \) were proper. Thus,

\[
k_r = (x - \lambda_r)F(x)|_{x=\lambda_r}
\]

For quadratic or repeated factors, the appropriate procedures discussed in Secs. B.5-2 or B.5-3 should be used as if \( F(x) \) were proper. In other words, when \( m = n \),
the only difference between the proper and improper case is the appearance of an extra constant $b_n$ in the latter. Otherwise the procedure remains the same. The proof is left as an exercise for the reader.

\section*{Example B.11}

Expand $F(x)$ into partial fractions if

$$F(x) = \frac{3x^2 + 9x - 20}{x^2 + x - 6} = \frac{3x^2 + 9x - 20}{(x - 2)(x + 3)}$$

Here $m = n = 2$ with $b_n = b_2 = 3$. Therefore,

$$F(x) = \frac{3x^2 + x - 20}{(x - 2)(x + 3)} = 3 + \frac{k_1}{x - 2} + \frac{k_2}{x + 3}$$

in which

$$k_1 = \frac{3x^2 + 9x - 20}{(x - 2)(x + 3)} \bigg|_{x=2} = \frac{12 + 18 - 20}{(2 + 3)} = \frac{10}{5} = 2$$

and

$$k_2 = \frac{3x^2 + 9x - 20}{(x - 2)(x + 3)} \bigg|_{x=-3} = \frac{27 - 27 - 20}{(-3 - 2)} = \frac{-20}{-5} = 4$$

Therefore,

$$F(x) = \frac{3x^2 + 9x - 20}{(x - 2)(x + 3)} = 3 + \frac{2}{x - 2} + \frac{4}{x + 3}$$

\section*{B.5-6 Modified Partial Fractions}

Often we require partial fractions of the form $\frac{kx}{(x - \lambda)^r}$ rather than $\frac{k}{(x - \lambda)^r}$. This can be achieved by expanding $F(x)/x$ into partial fractions. Consider, for example,

$$F(x) = \frac{5x^2 + 20x + 18}{(x + 2)(x + 3)^2}$$

Dividing both sides by $x$ yields

$$\frac{F(x)}{x} = \frac{5x^2 + 20x + 18}{x(x + 2)(x + 3)^2}$$

Expansion of the right-hand side into partial fractions as usual yields

$$\frac{F(x)}{x} = \frac{5x^2 + 20x + 18}{x(x + 2)(x + 3)^2} = \frac{a_1}{x} + \frac{a_2}{x + 2} + \frac{a_3}{(x + 3)} + \frac{a_4}{(x + 3)^2}$$

Using the procedure discussed earlier, we find $a_1 = 1$, $a_2 = 1$, $a_3 = -2$, and $a_4 = 1$. Therefore,

$$\frac{F(x)}{x} = \frac{1}{x} + \frac{1}{x + 2} - \frac{2}{x + 3} + \frac{1}{(x + 3)^2}$$

Now multiplying both sides by $x$ yields

$$F(x) = 1 + \frac{x}{x + 2} - \frac{2x}{x + 3} + \frac{x}{(x + 3)^2}$$

This expresses $F(x)$ as the sum of partial fractions having the form $\frac{kx}{(x - \lambda)^r}$. 