Laplace Transform Theorems:

LECTURE 9

1. <u>The Final Value Theorem</u>: If all poles of sY(s) are in the left half of the s-plane, then

$$\lim_{t \to \infty} y(t) = \lim_{s \to 0} sY(s).$$

<u>NOTE</u> i) if poles on the right side ⟹ unbounded
ii) if poles on the y-axis (imaginary) ⟹ oscilation.

2. <u>The Initial Value Theorem</u>: For any Laplace transform pair,

$$\lim_{s \to \infty} sF(s) = f(0^+).$$

Find the final value (as time goes to infinity!) of the following functions:

(i) $f(t) = e^{-t}$

(ii) $w(t) = 5(1-e^{-4t})$

(iii) $C(t) = \sin(2t)$

(ADAPTED FROM SATISH NAIR)

1) $f(t) = e^{-t}$ $\Rightarrow F(s) = \frac{1}{s+1}$ (1 pole @ -1)

So, $\lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{s}{s+1} = 0 = f(\infty)$

$\lim_{s\to\infty} sF(s) = 1 = f(0^+)$

2) $w(t) = 5(1-e^{-4t})$ $\Rightarrow W(s) = \frac{5}{s} - \frac{5}{s+4} = \frac{20}{s(s+4)}$

1 pole @ -4
1 pole @ 0

So, $\lim_{s\to 0} sW(s) = \lim_{s\to 0} \frac{20}{(s+4)} = \frac{20}{4} = 5 = w(\infty)$

$\lim_{s\to\infty} sW(s) = \frac{20}{\infty} = 0 = w(0^+)$

3) $c(t) = \sin(2t)$ $\Rightarrow C(s) = \frac{2}{s^2+4}$

1 pole @ -2
1 pole @ 2

$\Rightarrow$ can't apply FV theorem

$\lim_{s\to\infty} \frac{2s}{s^2+4} = 0 = c(0^+)$

$\rightarrow$ faster growth than $2s$

3.6 SOLUTION METHODS FOR DIFFERENTIAL EQUATIONS

Introduction

Application of the physical laws to a system usually yield ordinary differential equations (ODE). Two basic methods of solution are

- a) analytical
  - i) Laplace transform method
  - ii) Homogeneous (Complementary)/Particular solution method
- b) numerical

The analytical method is limited to linear equations with constant coefficients, while the numerical method can solve both linear and nonlinear equations.
General form of the equation considered:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_o x = f(t)$$

$a_i \equiv$ constant
$f(t) \equiv$ known function of time

Solution:
1) find the complementary part of the solution x_c (also called x_h),
2) find the particular solution x_p,
3) add x_c to x_p to get the total solution, <u>and then only</u> apply the initial conditions to evaluate the constants of integration.

Read text to find how to treat double/triple roots for the complementary part.

Principle of Superposition

The principle of superposition applies only to linear differential equations (need not have constant coefficients.)

If the particular solution to $f_1(t)$ is $x_{p1}(t)$ and to $f_2(t)$ is $x_{p2}(t)$ ...... and to $f_n(t)$ is $x_{pn}(t)$, then the particular solution to $[f_1(t)+f_2(t)+\cdots+f_n(t)]$ is $[x_{p1}(t)+x_{p2}(t)+\cdots+x_{pn}(t)]$.

Solve $\dot{x}+3x=1 \quad x(0)=1$

NOTE: $f^{(n)}(t) \Rightarrow s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$

$$\dot{x}(t)+3x(t)=1$$

$$\mathcal{L}\left(\dot{x}(t)+3x(t)\right) = \mathcal{L}(1)$$

$$\dot{x}(t) \Rightarrow sX(s) - x(0)$$

$$sX(s) - \cancel{x(0)}^{\,1} + 3X(s) = \frac{1}{s} \Rightarrow (s+3)X(s) = \frac{1}{s}+1$$

$$X(s) = \frac{s+1}{s(s+3)} \Rightarrow \frac{1/3}{s} + \frac{2/3}{s+3} \overset{\mathcal{L}^{-1}}{\Longrightarrow} \left(\frac{1}{3}+\frac{2}{3}e^{-3t}\right)1(t)$$

Partial Fraction expansion (cover-up)

Solve $\ddot{x} + 3\dot{x} + 2x = u \quad x(0) = 0;$ ***u is unit step***

$\dot{x}(0) = \alpha$

NOTE: $f^{(n)}(t) \Rightarrow s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{(n-1)}(0)$

$$s^2 X(s) - s\,x(0) - \dot{x}(0) + 3\left[s X(s) - x(0)\right] + 2X(s) = \frac{1}{s}$$

$$s^2 X(s) - s.0 - \alpha + 3s X(s) - 3.0 + 2X(s) = \frac{1}{s}$$

$$(s^2 + 2)\,X(s) = \frac{1}{s} + \alpha = \frac{\alpha s + 1}{s}$$

$$X(s) = \frac{\alpha s + 1}{s(s^2+2)} = \frac{?}{s} + \frac{?}{s + \sqrt{2}j} + \frac{?}{s - \sqrt{2}j}$$

Continue this ...

Solve:

$$\ddot{y} + 5\dot{y} + 6y = 2e^{t} + 5t \qquad \text{(i)}$$

initial conditions: $y(0) = 10$

$\dot{y}(0) = 0$

Answer: $s^2Y(s) - 10s + 5[sY(s) - 10] + 6Y(s) = \frac{2}{s-1} + \frac{5}{s^2}$

$$[s^2 + 5s + 6]\,Y(s) = \frac{2}{s-1} + \frac{5}{s^2} + 10s + 50$$

check the textbook for a similar example

3.8 BLOCK DIAGRAMS

Differential (and algebraic) equations can be expressed in block diagram notation using Laplace transforms.

e.g. $\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = 9u(t)$ [the input is u(t) and the output is x(t)]

Taking Laplace transforms with zero ICs (zero ICs are always assumed for transfer functions)

$$s^2 X(s) + 3sX(s) + 2X(s) = 9U(s)$$

$$\Rightarrow \frac{X(s)}{U(s)} = \frac{9}{s^2 + 3s + 2}, \quad \text{i.e., } X(s) = \frac{9}{s^2 + 3s + 2} * U(s)$$

This is represented in block diagram form as follows

$$\xrightarrow{U(s)} \boxed{\frac{9}{s^2 + 3s + 2}} \xrightarrow{X(s)}$$

Laplace transform of the output = Transfer Function of Dynamic System * Laplace transform of the input

Problem: Find the relation between input u(t) and output y(t) for the following dynamic system

$$\ddot{x}(t) + 8\dot{x}(t) + 9x(t) = 5u(t)$$
$$\dot{y}(t) + 2y(t) = 3x(t)$$

$$s^2 X(s) + 8sX(s) + 9X(s) = 5U(s)$$

$$sY(s) + 2Y(s) = 3X(t)$$

$$X(s)\left[s^2 + 8s + 9\right] = 5U(s) \Rightarrow X(s) = \frac{5}{s^2 + 8s + 9} U(s)$$

$$sY(s) + 2Y(s) = 3 * \frac{5}{s^2 + 8s + 9} U(s)$$

$$Y(s) = \frac{15}{(s+2)(s^2 + 8s + 9)} U(s)$$

Example of combining block diagrams – what do they mean?

Problem: Find the relation between input u(t) and output y(t) for the following dynamic system

$$\ddot{x}(t) + 8\dot{x}(t) + 9x(t) = 5u(t)$$
$$\dot{y}(t) + 2y(t) = 3x(t)$$

<u>Manipulation of equations</u>

From the second equation,

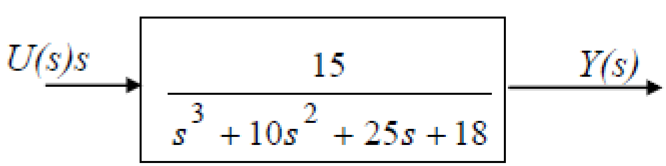
$$x(t) = \frac{\dot{y} + 2y}{3}$$

$$\Rightarrow \dot{x}(t) = \frac{\ddot{y} + 2\dot{y}}{3} \text{ and } \ddot{x}(t) = \frac{\dddot{y} + 2\ddot{y}}{3}$$

Substituting $\ddot{x}$, $\dot{x}$ and x into the first equation, we get

$$\dddot{y}(t) + 10\ddot{y}(t) + 25\dot{y}(t) + 18y(t) = 15u(t)$$

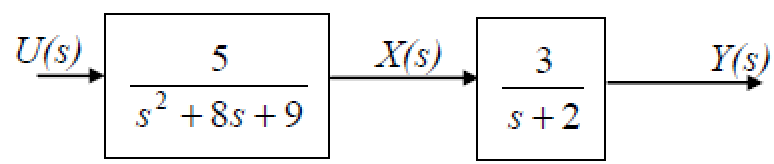
Taking Laplace transform of this equation, we get

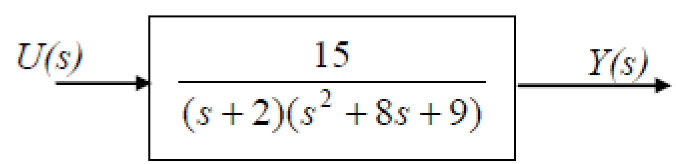


which is the same result as on the RHS.

<u>Block Diagram Method</u>

Taking Laplace transforms of each equation separately, we can develop the following block diagram structure



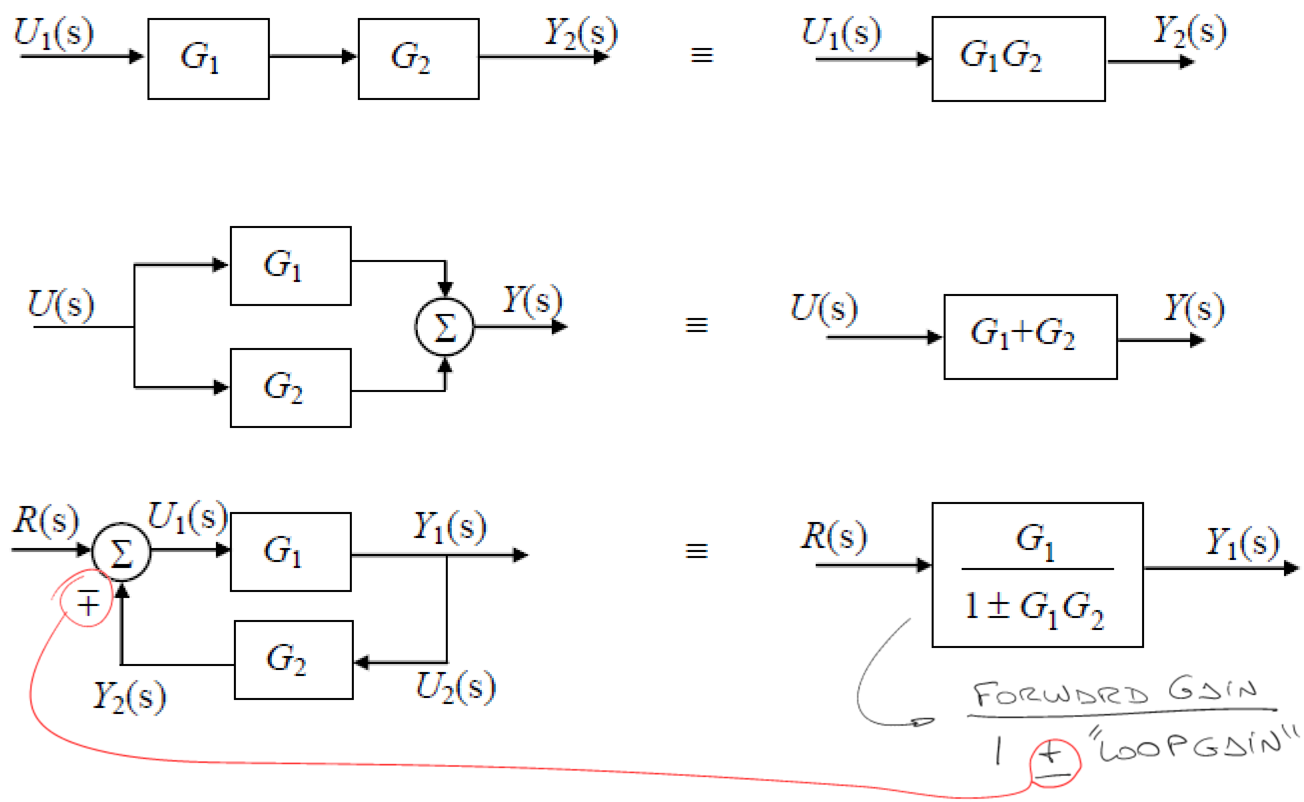


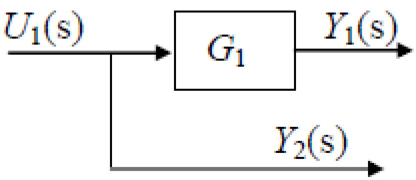
i.e., in this case just multiply the two TFs, so that the intermediate variable x is eliminated. This is the same as the result on the LHS. So this block diagram manipulation results in the same answer as the process as the process of manipulating equations, on the LHS.

Block Diagram Manipulation

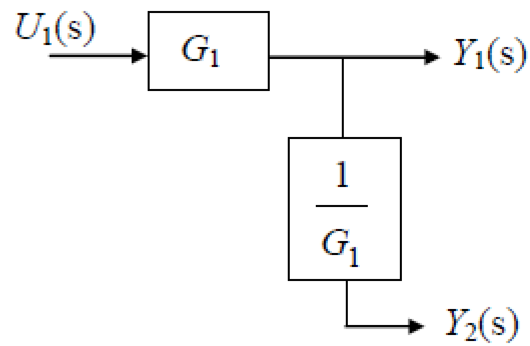
- The transfer function of each component is placed in a box, and the input-output relationships between components are indicated by lines and arrows. Solving the equations by graphical manipulation is often easier and more informative than algebraic manipulation.

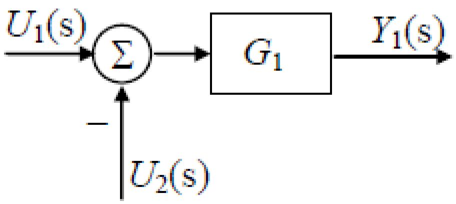
- Examples of block diagram algebra. (Note: $G(s)$ indicated by G)



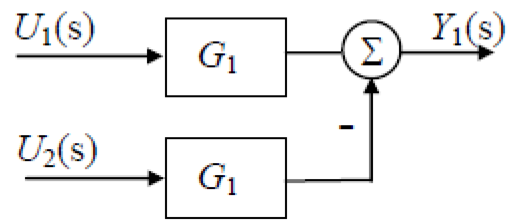

$U_1(s)$
G_1
$Y_1(s)$
$Y_2(s)$

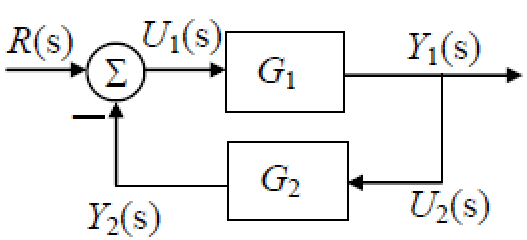
≡


$U_1(s)$
G_1
$Y_1(s)$
$\frac{1}{G_1}$
$Y_2(s)$

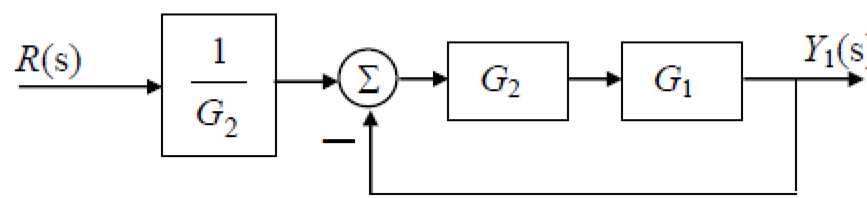

$U_1(s)$
Σ
G_1
$Y_1(s)$
$-$
$U_2(s)$

≡

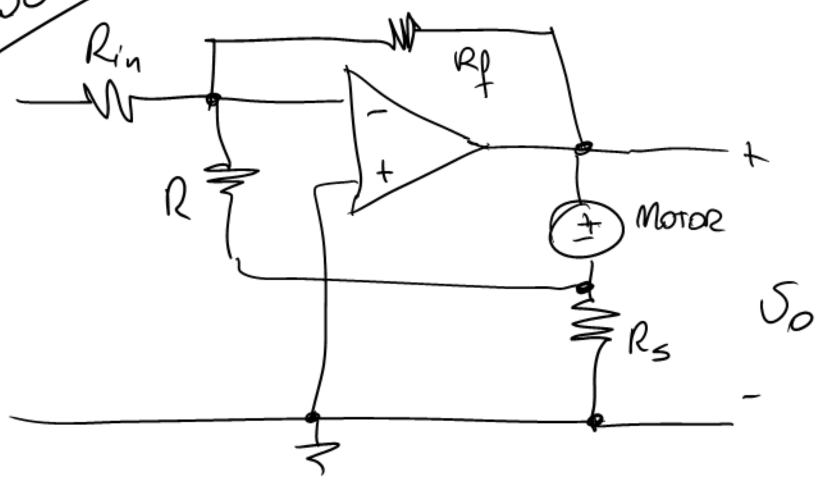

$U_1(s)$
G_1
Σ
$Y_1(s)$
$U_2(s)$
G_1
-

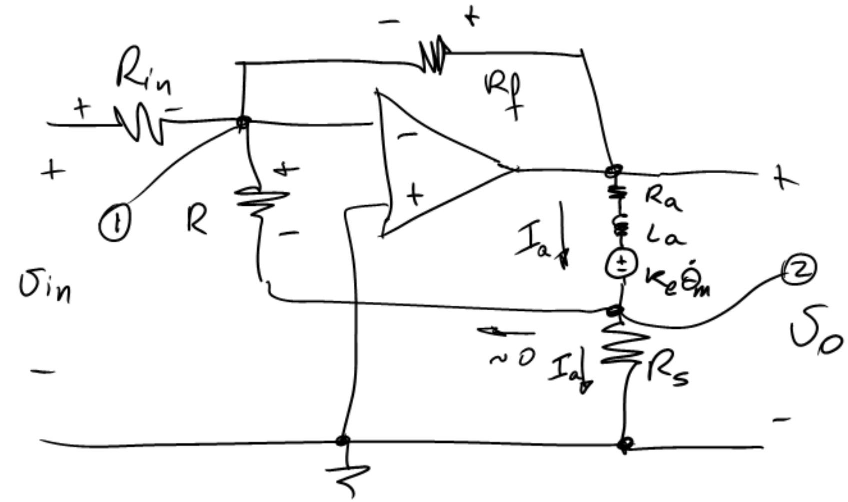

$R(s)$
Σ
$U_1(s)$
G_1
$Y_1(s)$
$-$
G_2
$Y_2(s)$
$U_2(s)$

≡


$R(s)$
$\frac{1}{G_2}$
Σ
G_2
G_1
$Y_1(s)$
$-$

Prob 2.13





KCL @ 1

$$\frac{v_{in} - v^-}{R_{in}} + \frac{v_o - v^-}{R_f} = \frac{v^- - v_z}{R} \qquad v^- = v^+ = 0$$

$$\frac{v_{in}}{R_{in}} + \frac{v_o}{R_f} = -\frac{v_z}{R}$$

KCL @ 2

$$I_a = \frac{v_z}{R_s} + \frac{v_z - v^-}{R} \Rightarrow I_a\left(\frac{R R_s}{R + R_s}\right) = v_z$$

$$v_o = I_a R_a + L_a \frac{dI_a}{dt} + K_e \dot{\Theta}_m + v_z =$$

$$V_{in}(s) + \frac{R_{in}}{R_f}\left[R_a + L_a s + K_e s\Theta(s) + \frac{RR_s}{R+R_s}\right]I_a(s) = -R_{in}\left(\frac{RR_s}{R+R_s}\right)\frac{I_a(s)}{R}$$

$$(*)\quad V_{in}(s) + \left\{\frac{R_{in}}{R_f}\left[R_a + L_a s + K_e s\Theta(s) + \frac{RR_s}{R+R_s}\right] + \frac{R_{in}R_s}{R+R_s}\right\}I_a(s) = 0$$

From pg 53 (eq 2.62) $\quad J_m\Theta(s)s^2 + b\Theta(s)s = K_t I_a(s)$

$$\Rightarrow \Theta(s) = \frac{K_t}{J_m s^2 + bs} I_a(s) \quad \text{in } (*)$$

$$\boxed{V_{in}(s) + \left\{\frac{R_{in}}{R_f}\left[R_a + L_a s + K_e \cancel{s}\frac{K_t}{J_m s^{\cancel{2}} + b\cancel{s}} + \frac{RR_s}{R+R_s}\right] + \frac{R_{in}R_s}{R+R_s}\right\}I_a(s) = 0}$$

if $R_s \ll R$ $\quad \frac{RR_s}{R+R_s} \cong R_s \quad \frac{R_s}{R+R_s} \simeq 0$

$$\frac{I_a(s)}{V_{in}(s)} = -\frac{R_f(J_m s + b)}{J_m R_{in} L_a s^2 + (R_{in}R_a J_m + R_{in}L_a b + J_m R_s)s + (R_{in}R_a b + K_e K_t + R_s b)}$$

If $R_f \to \infty$

$$V_{in}(s) + \left\{ \overbrace{\cancel{\frac{R_{in}}{R_f}\left[R_a + L_a s + K_e\left(\frac{K_t}{J_m s + b}\right) + \frac{R R_s}{R+R_s}\right]}}^{\to\, 0} + \frac{R_{in} R_s}{R+R_s} \right\} I_a(s) = 0$$

$$\frac{I_a(s)}{V_{in}(s)} = \frac{R + R_s}{R_{in} R_s}$$